Continued Fractions and abc-Triples

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What are continued fractions?

- Continued fractions are a way of expressing real numbers as a series of fractions.
- For a rational number, the continued fraction yields the exact fractional expression of the given number.
- My project dealt with continued fractions of irrational numbers and their relationship to high quality abc-triples.

What are convergents?

- To find each term in the continued fraction expansion of k, take $floor(k)$, $floor\left(\frac{1}{k-floor(k)}\right), \dots$
  - $floor\left(109^{\frac{1}{5}}\right) = 2, floor\left(\frac{1}{109^{\frac{1}{5}}-2}\right) = 1, \dots$
- The nth convergent is the fractional approximation given after taking this process to n steps.
- For irrational numbers, there is no end to this process (because there can be no exact fractional expression), so we are interested in the convergents.

What are abc-triples?

- $rad(x)$ is defined as a function that takes an integer x and removes any repeated factors
  - $rad(24) = 6,\ rad(30) = 30$
- The abc-conjecture states that for any positive integers a, b, c, with no common factors and $a + b = c$, that for all $\varepsilon > 0$, there are finitely many triples a, b, c, that satisfy $c < rad(abc)^{1+\varepsilon}$.
- For any abc triple, there is a number q such that $c = rad(abc)^q$, q is called the quality of the abc-triple

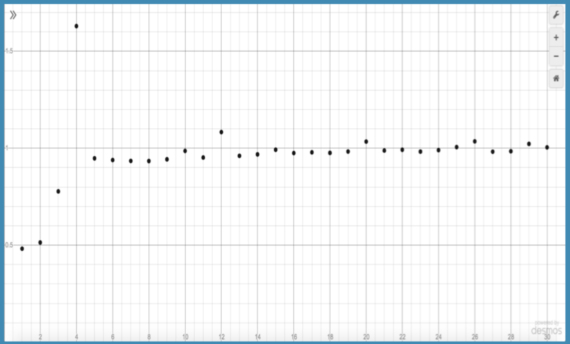
$$3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\ddots}}}}}}}}$$

The continued fraction of π to the 8th convergent

```
rmax=100000
nmax=25
imin=2
imax=5
bignum=1000
qmin=1.4
for r in range(2,rmax):
    # print "r=", r
    for n in range(2,nmax):
        # print "n=", n
        k=r^(1/n)
        if not k.is_integer():
            cf=continued_fraction(k)
            # print "cf", cf
            for i in range(imin,imax):
                if cf[i] > bignum:
                    # print cf[i]
                    d=cf.denominator(i-1)
                    e=cf.numerator(i-1)
                    c=max(r*d^n,e^n)
                    b=min(r*d^n,e^n)
                    a=c-b
                    if gcd(a,b)==1:
                        q = log(c)/log(radical(a)*radical(b)*radical(c))
                        if q > qmin:
                            print cf.convergent(i-1), r, n, q.numerical_approx()

23/9 109 5 1.62991168412705
7 2400 4 1.45567310017737
80/3 18963 3 1.44330674451084
23/3 26487 5 1.62991168412705
1573/120 29525 4 1.40720842445481
83/9 66708 5 1.40897279363458
```

The code I wrote to look for high quality abc-triples



A graph of quality abc triple given vs convergent for the first 30 convergents of $109^{1/5}$

nth convergent of $109^{1/5}$	nth term	abc-triple	Quality
2	2	$2^5+77=109$	0.48223
3	1	$109+134=3^5$	0.51395
5/2	1	$5^5+363=109*2^5$	0.77757
23/9	4	$109*3^{10}+2=23^5$	1.62991
1787864/699599	77733	$1787864^5+276921027679897l6067 =109*699599^5$	0.94804

How are these two things related?

- If the i^{th} term in a continued fraction of a $r^{1/n}$ is very large, then the $i-1^{th}$ convergent, $\frac{e}{d}$, of the continued fractions gives a very accurate approximation of $r^{1/n}$.
- $rd^n - e^n,\ rd^n,\ e^n$ make an abc triple!
- $rd^n - e^n$ is small, and $rd^n,\ e^n$ have lots of repeated factors, so we get a high quality abc-triple!

Why is this important?

- The abc-conjecture states that there are finite abc-triples with $q > 1+\varepsilon$.
- Empirical data shows that the abc-conjecture is likely true, with only 241 known abc-triples with $q > 1.4$, and 3 known with $q > 1.6$.
- If we continue finding high quality triples (or if we find no more high quality triples), we can get a better idea of the bound of ε and thus be one step closer to proving the abc-conjecture!

My research and topics *to be continued*... (budum pshh)

- I wrote a python code in CoCalc to find abc-triples using this method, compute their quality, and print all abc-triples found with $q > 1.4$.
- Run over a large range. this method found the 1st, 37th, 63rd, 191st, and 198th highest quality known triples.
- A topic for future research would be to find solutions to Pell's equation, $x^2 - Dy^2 = 1$, over a range of D and see if any of the resulting abc-triples are of high quality