The abc Conjecture

The abc conjecture claims that the sum of two numbers that factor a lot will not factor a lot.

\[ a + b = c \]

with \( c \geq a, b \) and \( \gcd(a, b) = 1 \)

For \( \epsilon > 0 \), there are only finitely many triples with quality \( > 1 + \epsilon \) where quality, \( q \), of abc is defined as

\[
q(a,b,c) = \frac{\log(c)}{\log(\text{rad}(abc))}
\]

i.e. \( c < \text{rad}(abc)^{1+\epsilon} \)

abc Conjecture

Let \( \epsilon > 0 \)

Then there are only finitely many abc triples with quality \( > 1 + \epsilon \), i.e., \( N(\gamma) < \text{rad}(N(a\beta\gamma))^{1+\epsilon} \) for all but finitely many triples

Definitions

Norm: For \( \alpha = a + bi \) in \( \mathbb{Z}[i] \), we define its norm by \( N(\alpha) = a^2 + b^2 = |\alpha|^2 \)

Prime Factorization:

Integers: For \( n \in \mathbb{Z} \geq 2 \) can be written uniquely \( n = p_1^{e_1} \cdots p_r^{e_r} \) where \( p_1 < p_2 < \cdots < p_r \) are primes and \( e_1, e_2, \ldots, e_r \in \mathbb{Z} \geq 1 \)

Gaussian integers: Every nonzero \( \alpha \) in \( \mathbb{Z}[i] \) can be written uniquely as \( \alpha = u \cdot \prod_{i=1}^r \mathcal{T}_i \cdot \mathcal{P}_i^{e_i} \) where \( u \) is a unit, each \( \mathcal{T}_i \) is a Gaussian prime in the upper right quadrant or the positive real axis, and \( \mathcal{P}_i \in \mathbb{Z} \geq 1 \)

Radical: For \( n \in \mathbb{Z} \geq 2 \) and \( \alpha \in \mathbb{Z}[i] \)

\[
\text{rad}(\alpha) = p_1 \cdots p_r
\]

\[
\text{rad}(\mathcal{T}) = \mathcal{T}_1 \cdots \mathcal{T}_r
\]

\( \alpha\beta\gamma \) Triple: Three nonzero Gaussian integers \( \alpha, \beta, \gamma \) such that

\[ \alpha + \beta = \gamma \]

\[ N(\gamma) \geq N(\alpha), N(\beta) \]

\[ \alpha = a + bi, \beta = c + di \]

\[ \gcd(a, b) = 1 \]

Quality: \( q = \log(N(\gamma)) / \log(N(\text{rad}(\alpha\beta\gamma))) \)

High Quality Hit: An \((\alpha, \beta, \gamma)\) triple with quality \( > 1 \)

Background

Gaussian integers are the subring of the complex numbers consisting of elements \( \alpha = a + bi \) where \( a, b \) in \( \mathbb{Z} \). They are mapped into the complex plane as is shown below:

References


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