Contents

1. Background and research questions
2. Portfolio allocation models
3. Data collection
4. Data examination
5. Training and testing the models
6. Results
7. Discussion and limitations
8. Conclusion
Context - Why Diversify?

- Imagine buying shares from a single company, X, with $500, and let’s say in one year, the stock has an equal chance of being worth $1000 or nothing.
  - Its variance, Var(X), would be 250 dollars^2, and its expected payout is $500.
- If instead we could buy two stocks, X and Y, each with $250, and each of which has an equal chance to be worth $500 or nothing in one year.
  - The variance of the portfolio after one year would be Var(X)+Var(Y)+2Cov(X,Y), which would equal at least zero, and at most 250 dollars^2, while the expected payout is $500.
  - If the two stocks were anything less than perfectly correlated, the portfolio’s variance would be less than the original portfolio’s while having the same expected payout.
- The basis of optimal portfolio allocation is this premise: there exists a line, given different companies’ returns and correlations, that defines the minimal risk a portfolio can take for any given return.
Background: Optimal Portfolio Allocation

- Given a set of financial assets (e.g. stocks) and a fixed investment amount (e.g. $1000), **in what proportion should we allocate our wealth towards buying each stock/asset in order to maximize our risk-adjusted returns?**\(^1\)
- Is buying every asset in **equal proportion**, and thereby spreading risk, the best approach?
- If not, how can we determine the **optimal proportions (weights)** with which we should buy each asset?
- We will examine real stock data to answer these questions.
  - We neglect transaction costs of making investments.
  - We use daily returns of stocks in our computations \(\frac{P_{\text{Day } i} - P_{\text{Day } i-1}}{P_{\text{Day } i-1}}\), where \(P\) is the closing price of a given stock.

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Portfolio Allocation Models

1. Markowitz Model

2. Interval Portfolio Model

3. Hierarchical Clustering Model
Markowitz Model

- In 1952, Harry Markowitz developed a method to find the ideal minimum-volatility portfolio for any given return.
- The tradeoff between risk and return can be expressed as a curve called the “optimal frontier” where for any given level of risk, returns are maximized, and vice versa, with a specific weighting of assets.¹
- The Markowitz Model assumes that investors are only allowed to go “long” or purchase assets.
  - In our version of the Markowitz model, we assumed that shares could also be shorted (negative returns are allowed).

Statistical parameters in the context of stocks

- The mean vector represents the average returns of the stocks in the data set.
- Variance represents the volatility of stocks: the relationship is that variance = volatility\(^2\)
- The vector of stock returns can be constructed as a vector of random variables \(X_1, X_2, \ldots\) such that each \(X_i\) is a normally-distributed random variable with the mean as the mean return of the stock and standard deviation as the volatility of the stock.
- We want to define a weighting of stocks such that for any return, we have the minimum volatility portfolio.
- Given a vector of weights that sum to 1, \(w\), how do we calculate the portfolio’s return and volatility?
  - \(w^T \mu\) = mean returns; we can set a desired return \(r\) such that \(w^T \mu = r\)
  - \(w^T \Omega w\) = variance = volatility\(^2\)
Markowitz Model Output

(Left) The “Markowitz Bullet” optimal frontier for our stock data (see data collection). The x-axis of the graph is the annual volatility of a portfolio, and the y-axis of the graph is the annual expected return of the portfolio. The point on the graph with the lowest volatility, denoted with a black dot and intersecting line, denotes the return (13.54%) that is obtainable with the lowest associated risk. This is the optimal portfolio as determined by the Markowitz model.
Portfolio Allocation Models

1. Markowitz Model

2. Interval Portfolio Model

3. Hierarchical Clustering Model
Interval Model

- The interval model is similar to the Markowitz Model, except instead of aiming to find the optimal frontier, we aim to maximize the probability of achieving a range of returns.
- We specify a lower bound, \( r \), and an upper bound, \( R \), of returns and calculate, using the Markowitz model, the relative probability of either occurring. Then, we can use optimization techniques to find the weighting required to achieve said returns.\(^1\)

Interval Model Continued

The return of optimum-weighted portfolio will return $w^T X$, where $X$ is the vector of normally distributed random variables representing returns, and $w$ represents the vector of weights. To get the probability that the returns fall in the interval, we can find $P(r < w^T X < R)$. Given that we assume the return random variables are normally distributed, we know that the return of the portfolio must also be normally distributed.

Therefore, by standardizing the return, we get the following:

$$P(r < w^T X < R) = P(R < w^T X) - P(r < w^T X) = \Phi((R - w^T \mu)/sd) - \Phi((r - w^T \mu)/sd)$$

Where $sd$ is the standard deviation of $w^T X$, which is equal to $\sqrt{w^T \Omega w}$.
Portfolio Allocation Models

1. Markowitz Model

2. Interval Portfolio Model

3. Hierarchical Clustering Model
Hierarchical Clustering Model

Intuition:
- Some stocks may be close substitutes of one another, i.e. they may have similar trends in returns over time.
- We expect companies in the same industry to have correlated performance, and we want to diversify not just across companies, but across industries.\(^2\)

Implementation:
- Ultrametric distance:\(^3\) \(\sqrt{2(1 - \rho)}\)
- Average linkage for hierarchical clustering
- Output is a vector of weights for our model
- This model aims to broaden exposure to all industries by allocating weights equally within clusters.


Hierarchical Clustering Output

(Above) The dendrogram produced through hierarchical clustering of our data. Each branch represents one stock; we use the branching here to calculate stock weights as described on the previous slide.
Data collection

- We retrieved five years of daily S&P 500 stock prices from 2013 to 2018.
  - Taken from Yahoo! Finance
- We then dropped all NAs, leaving us with 468 stocks.
- Then, we calculated daily returns for each stock.

(Above) First 20 rows and 11 columns of our stock dataframe in R - actual dimensions are 468 x 1260
Correlation Heat Maps

- We created correlation heat maps of the stock data to determine if there were any interesting relationships between certain stocks.
- We used the ggplot2 package to help create the heat maps.
- We apply hierarchical clustering to group the stocks on the heat map to get a better visualization of correlation within and across clusters.
- We also create a partial correlation heat map to remove the influence of other stocks when computing correlations.
Correlation Heat Map

- Most stocks exhibit a strong positive correlation (red), while many others exhibit a strong negative correlation (green).
- Few stocks exhibit little to no correlation (blue).
Partial Correlation Heat Map

- After removing the influence of other stocks, nearly all stocks exhibit little to no correlation (blue).
- Note that this is only representative of individual stocks, not industries.
To examine the distribution of the stocks in our data set, we use QQ-plots under the assumption that the returns of each stock follow a normal distribution. We examine the daily returns within stocks. A random sample of 48 stocks is selected from our data set. Some of the stocks appear slightly skewed, but overall we can make the normal assumption for our models.
Training Markowitz Bullet and Interval Portfolio Models

- Essentially, this required finding $\mu$ and $\Omega$, as they were the parameters that are used to construct the values in $w$.
- Vector $\mu$ was estimated by averaging daily returns over six month training period
- Matrix $\Omega$ was estimated by finding the covariance matrix of the returns over the same period
- We then used the $\mu$ and $\Omega$ matrices as the mathematical basis of the models.
Testing Each Model

- Rolling test windows of **6 months, 2 years, and 3 years**.
  1. Train models on **rolling 6 month** period, test on **rolling 6 mo. windows** (slides 28/29).
  2. Train models on **rolling 2 year** period, test on **rolling 2 year windows** (slides 30/31).
  3. Train models on **rolling 1 year** period, test on **rolling 3 year windows** (slides 32/33).
  → For each window, we test the models **once without rebalancing** and **once with rebalancing** at the halfway point.

- Train on **6 months**, test on following **3.5 years** - rebalance after various equally-spaced intervals (0-10 times) (slide 34).

- We use a starting investment of $100,000 in each case.
Testing Each Model Continued

- To elaborate, the “rolling” window means simply that we train on a certain number of months from the start of the data, then test on the following months to see how the weights we learned from the first six perform monetarily at the end of the six months. Then we “roll” by moving our training and testing period forward by one day.
- Once we have rolled through the whole data set, we have several data points, or “returns” that are the sum of the returns from each stock from the end of each testing period.
- We do this for each model, and then analyze the returns over time.
- We select 30 stocks at random from our data set to construct our portfolios.
Assessing Each Model

- We compare each of the three previous models to the most simple weight allocation model: the **equal weights model**.
  - Each stock is allocated an equal weight of $1/N$, where $N =$ the number of stocks in our data set.
- We examine the **standard deviations** of the returns at the ends of each testing period for each model.
- We conduct a **difference-in-means paired t-test** to compare the returns at the ends of each testing period of each of our three models to the baseline equal weights model.
6-Month Rolling Window (No Rebalancing):

Smoothing Function Applied For Legibility, Actual Returns in Background

Performance by Model on 2013-2018 Stock Prices
No Rebalancing
6-Month Rolling Window (Rebalance Halfway):

Smoothing Function Applied For Legibility, Actual Returns in Background

Performance by Model on 2013-2018 Stock Prices
Rebalancing
2-Year Rolling Window (No Rebalancing):

Smoothing Function Applied For Legibility, Actual Returns in Background

Performance by Model on 2013-2018 Stock Prices
No Rebalancing
2-Year Rolling Window (Rebalance Halfway):

Smoothing Function Applied For Legibility, Actual Returns in Background

Performance by Model on 2017-2018 Stock Prices
Rebalancing

- Equal Allocation
- HCA
- Interval Portfolio
- Markowitz Bullet
3-Year Rolling Window (No Rebalancing):

Smoothing Function Applied For Legibility, Actual Returns in Background

Performance by Model on 2017-2018

Stock Prices

No Rebalancing

Model
- Equal Allocation
- HCA
- Interval Portfolio
- Markowitz Bullet

Dates (2013 on)
3-Year Rolling Window (Rebalance Halfway):

Smoothing Function Applied For Legibility, Actual Returns in Background

Performance by Model on 2017-2018 Stock Prices
Rebalancing
Static Window, Entire Dataset (Rebalancing):

Reallocation Effect on Total Returns

Performance by Model
Using Different Numbers of Reallocations

Model
- Equal Allocation
- HCA
- Interval Portfolio
- Markowitz Bullet
# Standard Deviation of Models

<table>
<thead>
<tr>
<th></th>
<th>6 Months No Rebal.</th>
<th>6 Months Rebal.</th>
<th>2 Years No Rebal.</th>
<th>2 Years Rebal.</th>
<th>3 Years No Rebal.</th>
<th>3 Years Rebal.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>HCA</strong></td>
<td>7910.5</td>
<td>7775.5</td>
<td>17407.9</td>
<td>16228.9</td>
<td>16732.8</td>
<td>15440.3</td>
</tr>
<tr>
<td><strong>Markowitz Bullet</strong></td>
<td>10052.3</td>
<td>9879.5</td>
<td>10483.5</td>
<td>9424.4</td>
<td>9778.9</td>
<td>10959.6</td>
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<tr>
<td><strong>Interval Portfolio</strong></td>
<td>10014.3</td>
<td>9838.1</td>
<td>11135.6</td>
<td>10031.6</td>
<td>9987.1</td>
<td>11253.4</td>
</tr>
<tr>
<td><strong>Equal Weights</strong></td>
<td>6740.3</td>
<td>6678.0</td>
<td>11116.6</td>
<td>10729.6</td>
<td>4526.5</td>
<td>4510.6</td>
</tr>
</tbody>
</table>
Standard Deviations Across Tests: 6 Months Test

Standard Deviations For 6 Months, No Reallocation
Markowitz Bullet is highest

Standard Deviations For 6 Months, With Reallocation
Markowitz Bullet is highest
Standard Deviations Across Tests: 2 Years Test

**Standard Deviations For 2 Years, No Reallocation**
HCA is highest

**Standard Deviations For 2 Years, With Reallocation**
HCA is highest
Standard Deviations Across Tests: 3 Years Test

**Standard Deviations For 3 Years, No Reallocation**
HCA is highest

**Standard Deviations For 3 Years, With Reallocation**
HCA is highest
Significance Analysis

- Using a difference-of-means analysis via a paired t-test, we saw that most of the returns were significantly different from the baseline equal allocation portfolio. The chart below displays the t-values for the t-test.

- Due to the large data sets, we can approximate a distribution using a normal distribution, which, for an alpha of .01, leads to a two-sided critical z-score of 2.575. The significantly different portfolios are highlighted below in yellow.

<table>
<thead>
<tr>
<th></th>
<th>3 Yr No Reallocation</th>
<th>3 Yr Reallocation</th>
<th>2 Yr No Reallocation</th>
<th>2 Yr Reallocation</th>
<th>6 Mo No Reallocation</th>
<th>6 Mo Reallocation</th>
</tr>
</thead>
<tbody>
<tr>
<td>HCA</td>
<td>-11.7</td>
<td>-10.91</td>
<td>-23.22</td>
<td>-25.43</td>
<td>4.78</td>
<td>4.6</td>
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<tr>
<td>Markowitz</td>
<td>22.592</td>
<td>15.01</td>
<td>-1.52</td>
<td>-14.054</td>
<td>-2.19</td>
<td>-3.03</td>
</tr>
</tbody>
</table>
Results Summary

- There is a small but significant difference between models’ performances in 6 month rolling window.
- HCA performance drops at 2 year rolling window, but picks up greatly at 3 year rolling window.
  - Possibly HCA invests more in one industry that experiences an uptick in stock values towards the end of the 3-year window.
- Significant difference between HCA and Equal Allocation vs Interval Portfolio and Markowitz Bullet after testing for 3 years.
- Rebalancing decreases the variance of the models slightly, which is what one would expect.
Discussion

- In an investment scenario, it is helpful to rebalance your portfolio to reduce risk. However, there appears to be little benefit to rebalancing more often than once to several times a year (see graph on slide 32).
- The purpose of the Markowitz and Interval models is not necessarily to maximize returns, but to minimize risk. This was achieved in the two-years testing period, but not in the other periods, suggesting that the predictive power of historical data was not particularly high in this time period.
- From our two and three year runs, we can see that Markowitz and Interval Portfolio both produce much lower volatility overall than Hierarchical Cluster Analysis, which makes sense since HCA does not seek to minimize volatility.
Discussion Continued

- The HCA, which ascribes equal allocation to the clusters, exhibited more long-term variance, as it should since it does not, by itself, try to minimize covariance, but rather broadens exposure to different industries.
- Lastly, as a brief validation, the Markowitz model tended to have a smaller variance than the interval model, suggesting that the indeed, the relationship between returns and volatility held true in most cases.
Limitations of our Analyses

- We neglect transaction costs of purchasing assets
  - There is often a cost associated with re-allocating portfolios, so investing in a re-allocated portfolio could decrease returns.

- While the Markowitz and interval models are very strong at optimizing risk-adjusted returns under specific conditions, they assume stock price changes are normally distributed and that each day’s returns is independent of each other day’s returns.
  - In the long run, the latter assumption may be true, but the first assumption fails to take into account significant short-term outlier events such as Trump’s election or COVID-19, which can skew the results of the 6-month tests.
  - The Markowitz model assumes that past returns are indicative of future returns (and thus volatility). If we train the model on a particularly turbulent/calm period in the markets, it misallocates the stocks in the future.
  - The QQ-plots indicate the long-run normality of the data.
Limitations Continued

- In general, the hierarchical clustering model performed significantly worse than the other models. However, we do not calculate the risk-adjusted returns, which according to Raffinot (2017) should illustrate that the HCA model performs better than illustrated in our analyses. It should be noted that the HCA model may sacrifice a higher volatility for higher returns, while the Markowitz and interval models seek to minimize volatility.

- To construct our portfolios, we randomly selected 30 stocks from our larger data set, as it is impractical to have a portfolio with 400+ stocks. However, it is quite possible our randomly selected portfolio had outlier stocks that did not perform similarly relative to the rest of the market (e.g. Netflix, Amazon, GE). Had we tested on a different random sample containing a different selection of stocks, our results may have been different.
  - If we had more time, it would have been interesting to compare our results for multiple random samples.
Concluding remarks

1. Rebalancing the portfolio reduces variance of all models.
2. Markowitz Bullet and Interval Portfolio perform better over long periods of time.
   - Little difference between models over a short period of time.
   - This makes sense, as the goal is to have a stable, long-term portfolio.
3. Limitations in dataset analysis and our models could impact results.
Thank you Professor Demidenko and M70!

“Mathematics is the queen and statistics is the king of all sciences”

- Professor Demidenko

Questions?