The de Bruijn sequence is an example of a more general concept: a universal cycle. These are cyclic sequences of length $n$ where each consecutive group of length $k$ is a unique object. The object for each cycle could be different:

- **Binary strings**
  - We applied this in the de Bruijn sequences
  - Subsets
    - Subsets of size $k$ of an $n$-element set
    - Ex: subsets of size 2 from the set $\{0,1,2,3,4\}$

- **Permutations**
  - Different orderings of $k$ elements
  - Ex: all possible orderings of 3 numbers, highest (3) to lowest (1)

- **Set Partitions**
  - A set of $n$ elements can be arranged into different groups
  - Ex: The partitions of the set $\{A, B, C\}$

All together: $(A, B, C)$
Two together: $(A)(B, C) \cup (A, B)$
All apart: $(B)(A, C)$

---

**Applications of Universal Cycles**

**Magic**

**Overall goal:** Have three audience members pick out the top three cards and be able to guess those cards.

**How to get there:**

- Take the de Bruijn cycle we showed previously (1,1,0,1,0,0,0,1)
- Let 1 correspond to a red card, and 0 correspond to a black card
- For each of the elements in the sequence, assign a card that matches the color
  - 1: red
  - 0: black

- Because of its cyclic nature, the order is maintained even if you ask the audience to cut the cards
- Have three audience members pick the top three cards, ask them to raise their hands if they have a red card
- This gives the ordering of red and black cards, which indicates where in the deck the audience has removed their cards from

**How to find a De Bruijn Cycle of window length $n$:**

- Create a graph where:
  - Each node is a possible binary string of length $(n-1)$
  - An edge goes from $x$ to $y$ if there is a binary string of length $n$ that has $x$ as its left and $y$ as its right
  - Follow the edges until you have used each edge only once and you end up where you started (Eulerian circuit)
- The graph has an Eulerian circuit because each vertex has an equal number of edges leading in and leading out

**De Bruijn sequences can be used for robots to be able to detect where they are in space. Therefore, instead of just focusing on a one-dimensional string, we apply the concepts of a universal cycle to a two-dimensional problem (de Bruijn array).**

- A de Bruijn array with window size $u \times v$ is an array of zeros and ones such that every $u \times v$ window of zeros and ones appears exactly once going around the edges.
- If the robot has the information in the given window, it can determine where in the grid it is located
- Applied in digital pens:
  - The paper has an invisible de Bruijn array printed on it
  - The pen's infrared camera detects the pattern and can determine where it is on the page

**Robotics**

De Bruijn sequences can be used for robots to be able to detect where they are in space. Therefore, instead of just focusing on a one-dimensional string, we apply the concepts of a universal cycle to a two-dimensional problem (de Bruijn array).

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**References**


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