Mathematics in Magic: Looking Into the Magic of Universal Cycles

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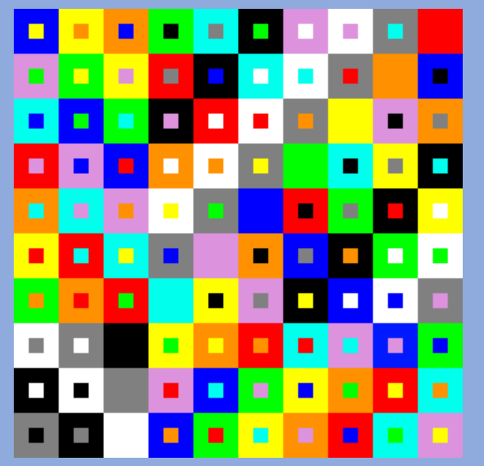

DE BRUIJN CYCLES

A de Bruijn sequence of order n on a size-k alphabet A is a cyclic sequence in which every possible length-n string on A occurs exactly once as a substring.

Ex: binary strings of length 3

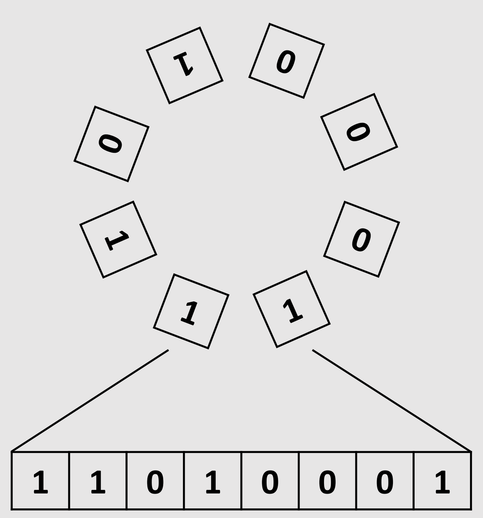

Figure 1. De Bruijn sequence with binary strings of length 3

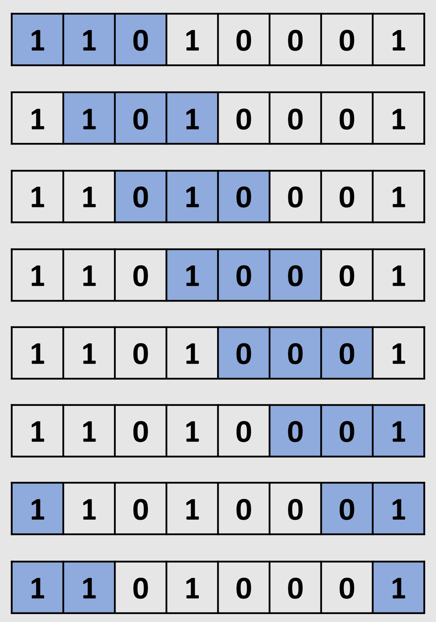

Figure 2. Illustration of all possible windows of length 3 within the cycle

- Each window of size 3 is a distinct binary string
- All possible binary strings of length 3 are present in the cycle
- If you know the values of three consecutive cells, you know your position within the cycle

UNIVERSAL CYCLES

The de Bruijn sequence is an example of a more general concept: a *universal cycle*. These are cyclic sequences of length n where each consecutive group of length k is a unique object. The object for each cycle could be different:

- Binary strings
  - We applied this in the de Bruijn sequences
- Subsets
  - Subsets of size *k* of an *n*-element set
  - Ex: subsets of size 2 from the set {0,1,2,3,4}

0	1	2	3	4	0	2	4	1	3

- Permutations
  - Different orderings of *k* elements
  - Ex: all possible orderings of 3 numbers, highest (3) to lowest (1)

123

1	4	6	2	5	3

231

- Set Partitions
  - A set of *n* elements can be arranged into different groups
  - Ex: The partitions of the set {A, B, C}

All together:	Two together:	All apart:
{A, B, C}	{A}{B, C} {C}{A, B} {B}{A, C}	{B}{A}{C}

APPLICATIONS OF UNIVERSAL CYCLES

Magic

Overall goal: have three audience members pick out the top three cards and be able to guess those cards.

How to get there:

- Take the de Bruijn cycle we showed previously (1,1,0,1,0,0,0,1)
- Let 1 correspond to a red card, and 0 correspond to a black card
  - The sequence becomes RRBRBBBR
- For each of the elements in the sequence, assign a card that matches the color

A H	2 D	A S	5 D	8 S	Q C	K S	K H

- Because of its cyclic nature, the order is maintained even if you ask the audience to cut the cards
- Once three audience members pick the top three cards, ask them to raise their hands if they have a red card
- This gives the ordering of red and black cards, which indicates where in the deck the audience has removed their cards from

How to find a De Bruijn Cycle of window length n:

- Create a graph where:
  - Each node is a possible binary string of length (n-1)
  - An edge goes from *x* to *y* if there is a binary string of length *n* that has *x* at its left and *y* at its right
- Follow the edges until you have used each edge only once and you end up where you started (Eulerian circuit)
- The graph has an Eulerian circuit because each vertex has an equal number of edges leading in and leading out

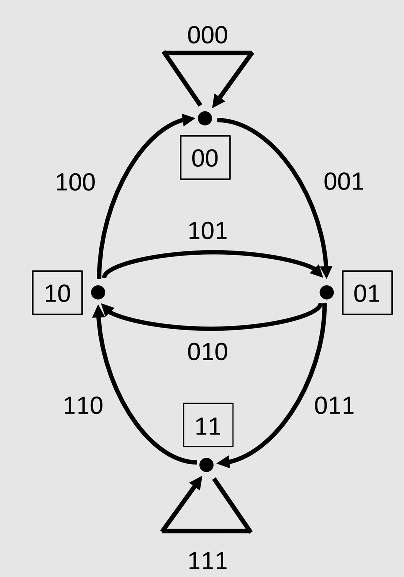

Figure 3. The de Bruijn graph on four vertices

Robotics

De Bruijn sequences can be used for robots to be able to detect where they are in space. Therefore, instead of just focusing on a one-dimensional string, we apply the concepts of a universal cycle to a two-dimensional problem (de Bruijn array).

- A de Bruijn array with window size u × v is an array of zeros and ones such that every u × v window of zeros and ones appears exactly once going around the edges.
- If the robot has the information in the given window, it can determine where in the grid it is located

1	1	0	1
0	0	0	1
1	0	0	0
1	0	1	1

1	1
0	0

Figure 3. De Bruijn array with window size 2 × 2

- Applied in digital pens:
  - The paper has an invisible de Bruijn array printed on it
  - The pen's infrared camera detects the pattern and can determine where it is on the page

MODIFIED MAGIC WITH SET PARTITIONS

We developed a method similar to the one outlined before but using set partitions to create universal cycles rather than binary strings.

We show below the process of finding a cycle with windows of length 3:

- To find the universal cycle, we used a graph, comparable to the one constructed with the de Bruijn sequence (on the right)
  - The numbers indicate the cards in 1^{st}, 2^{nd} and 3^{rd} positions respectively
- 1^{st} attempt to find the cycle:

A	A	B	A	C	C	C

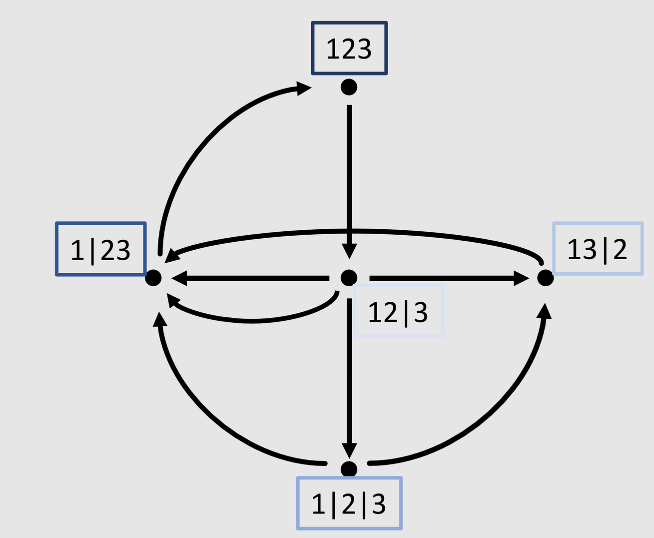

Figure 5. The partitions graph with 5 nodes (follow the nodes light to dark)

- We were unable to find a cycle that traversed each node exactly once (Hamiltonian cycle), so we tried one that would traverse each node twice:

A	A	B	A	C	C	C	B	C	A

- We applied the same logic to find a cycle with window length 4, with each letter corresponding to a different suit in a deck of cards
- A paper by Higgins et al. proved that there was a universal cycle for partitions of 4 elements, however, we were unable to find it, so we found one that cycled through the graph twice

A C	A S	A D	2 D	A H	2 H	3 D	3 H	4 D	2 S	5 D	6 D	7 D	8 D	2 C
A	B	C	C	D	D	C	D	C	B	C	C	C	C	A

Q H	Q D	6 S	7 S	J C	Q C	8 S	K C	9 S	K D	10 S	J S	Q S	K S	K H
D	C	B	B	A	A	B	A	B	C	B	B	B	B	D

Figure 6. The final sequence in the modified magic trick

REFERENCES

[1] Diaconis, P., & Graham, R. L. (2016). *Magical mathematics: The mathematical ideas that animate great magic tricks*. Princeton University Press.
[2] Higgins, Z., Kelley, E., Sieben, B., & Godbole, A. (n.d.). *Universal and Near-Universal Cycles of Set Partitions*. the electronic journal of combinatorics, 2015

ACKNOWLEDGMENTS

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