The Chromatic Symmetric Function in the Star Basis

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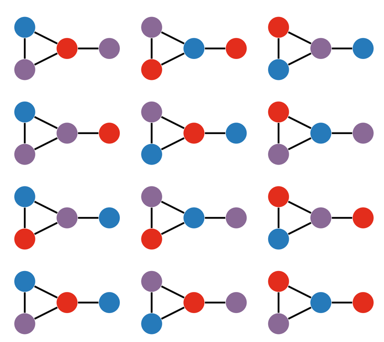
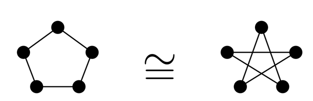
Background and Definitions

- A graph $G = (V, E)$ is defined by a set of vertices $V = V(G)$ and edges $E = E(G)$.
- In a **proper coloring** of a graph, two adjacent vertices have different colors.
- The **chromatic polynomial** of a graph $\chi(G, k)$ gives the number of proper colorings of the graph G with k colors.



$\chi(G, 3) = 12$

- The two graphs below are **isomorphic**:



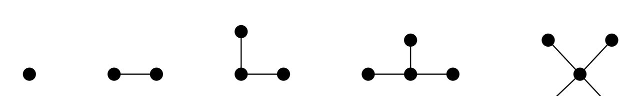
- An **internal edge** is an edge uv such that $\deg(u), \deg(v) \geq 2$
- The **internal degree** of a vertex v is the number of internal edges incident to v.
- A **leaf component** of a forest F is any maximal subtree $C \subseteq F$ such that every edge in C is a leaf-edge in F.
- In 1994, Stanley defined the **chromatic symmetric function** (CSF) of a graph as a generalization of its chromatic polynomial:

$$X_G = \sum_{\text{proper } \kappa: V \to \mathbb{N}} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots,$$

  where $x_1, x_2, \ldots$ are commuting variables.
- **Stanley's Tree Isomorphism Conjecture**: If T_1, T_2 are non-isomorphic trees, then $\mathbf{X}_{T_1} \neq \mathbf{X}_{T_2}$

The CSF in the star basis

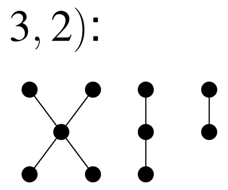
- A **star graph** on n vertices $St(n)$ is a tree with $n-1$ leaf vertices and one vertex of degree $n-1$.



- For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we define the star forest indexed by λ as follows:

$$St_\lambda = St_{\lambda_1} \cup \cdots \cup St_{\lambda_k}$$

- For example, for $\lambda = (5, 3, 2)$:



- Denote $X_{St_n} = \mathfrak{st}_n$. There is a well-known formula that expresses $\mathfrak{st}_n$ in terms of the power-sum symmetric polynomials:

$$\mathfrak{st}_{n+1} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} p_{(r+1, 1^{n-r})}$$

- Similarly as above, for $\lambda = (\lambda_1, \ldots, \lambda_k)$, we have:

$$\mathfrak{st}_\lambda = \mathfrak{st}_{\lambda_1} \cdots \mathfrak{st}_{\lambda_k}$$

- $\{\mathfrak{st}_\lambda : \lambda \vdash n\}$ is a basis for the algebra of symmetric functions of degree n.

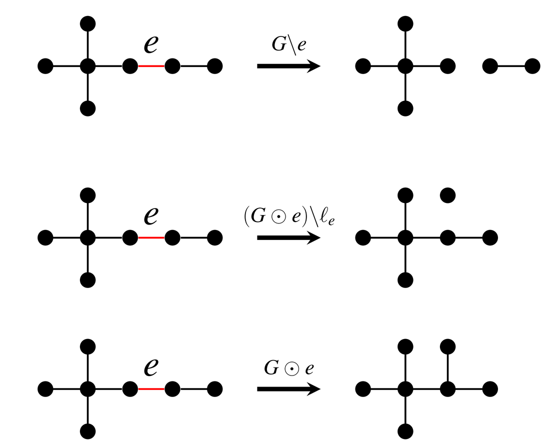
Deletion-Near-Contraction (DNC)

- We can write the CSF compactly using the star-basis:

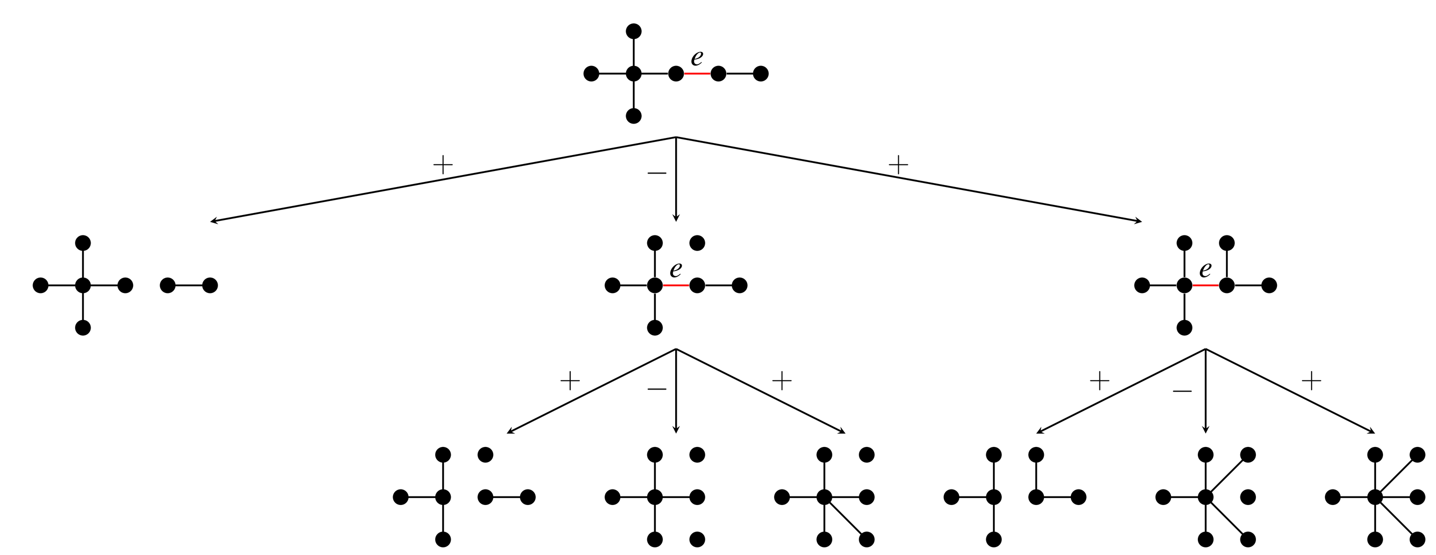
$$\mathbf{X}_G = \sum_{\lambda \vdash n} c_\lambda \mathfrak{st}_\lambda$$

- The DNC relation expresses the CSF of a forest recursively as a linear combination of CSFs of three forests, each with fewer internal edges [2]:

$$\mathbf{X}_G = \mathbf{X}_{G \setminus e} - \mathbf{X}_{(G \odot e) \setminus \ell_e} + \mathbf{X}_{G \odot e}$$



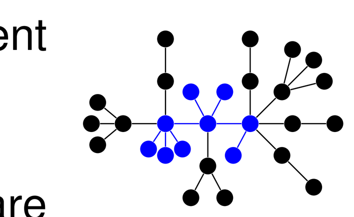
- This relation allows us to compute the CSF efficiently.
- Once we fix a permutation of the internal edges of a tree, we can visualize the operations performed in the DNC algorithm using a ternary tree:



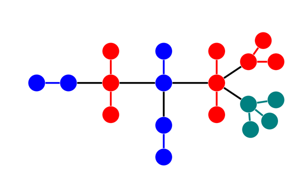
$$\mathbf{X}_G = \mathfrak{st}_{(7)} - 2\mathfrak{st}_{(6,1)} + \mathfrak{st}_{(5,2)} + \mathfrak{st}_{(5,1,1)} + \mathfrak{st}_{(4,3)} - \mathfrak{st}_{(4,2,1)}$$

Diameter Results

Let $\{v_1, \ldots, v_k\}$ be the set of internal vertices of a tree T whose internal degree is strictly greater than one. Let L_i be the set of leaf vertices adjacent to v_i.



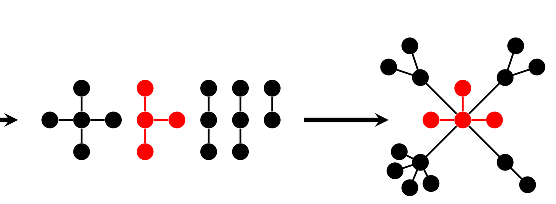
- The **internal subgraph** of T is the subgraph $I \subseteq T$ induced by $\{v_1, \ldots, v_k\} \cup L_1 \cup \cdots \cup L_k$. The **orders of the leaf components** in I are equal to $|L_1| + 1, \ldots, |L_k| + 1$
- For a partition λ of length $|I(T)|$ containing no 1's, we define the **edge-adjacency multiset** in which each part $p \in \lambda$ appears with multiplicity $m_{E_\lambda} = \max(m_{\lambda_\ell}(p) - m_\lambda(p), 0)$



- $c_{(5,4,3,3,2,2)} = 3 \implies \{3, 2\}^3$
- $c_{(4,4,3,3,3,2)} = 1 \implies \{2, 2\}$
- $c_{(6,4,3,2,2,2)} = 1 \implies \{3, 3\}$
- $c_{(7,3,3,2,2,2)} = 1 \implies \{4, 3\}$

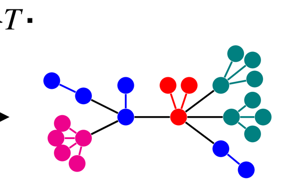
- **Theorem 1**: If T is any tree of diameter at most five, then the orders of the leaf components in the internal subgraph $I \subseteq T$ can be reconstructed from $\mathbf{X}_T$.
- **Corollary 1**: If T is a tree of diameter four, then T can be reconstructed from $\mathbf{X}_T$.

$\lambda_\ell = (5, 4, 3, 3, 2) \longrightarrow$



- **Corollary 2**: If T is a tree of diameter five such that the orders of the leaf components in I are distinct, then T can be reconstructed from $\mathbf{X}_T$.

$\lambda_\ell = (5, 4, 4, 3, 2, 2, 2)$
$\{2, 2\}, \{5, 2\}, \{3, 2\}^2, \{4, 3\}^2 \longrightarrow$



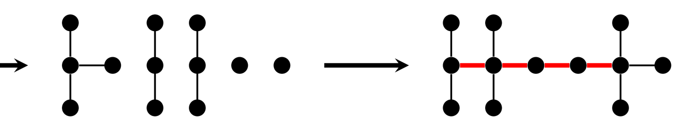
- **Theorem 2**: If T is a tree of diameter five such that the orders of the leaf components in I are equal, then T can be reconstructed from $\mathbf{X}_T$ by extending edge-adjacency multisets to partitions of length $|I(T) - 1|$.

Leading Partition

A CSF has **leading partition** λ if $c_\lambda \neq 0$ and $c_\mu = 0$ for all μ with larger lexicographic ordering. Given any $\lambda \vdash n$, we say λ is a **hook partition** of n if $n \geq 2$ and for some $1 \leq k \leq n-1$, $\lambda = (k, 1^{n-k})$.

- **Lemma 1:** Given any non-hook partition of n, λ, *Build-Leading-Tree* returns a tree on n vertices with leading partition λ.

$(4, 3, 3, 1, 1) \longrightarrow$



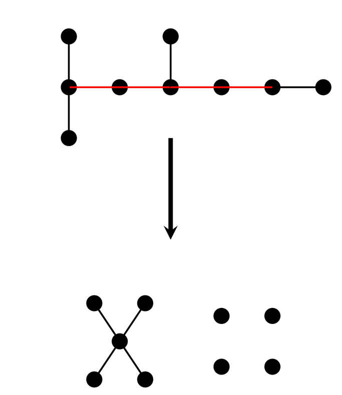
- **Lemma 2:** *Get-Leading-Partition* returns the leading partition of F.

Algorithm 1: Get-Leading-Partition
input : any forest F
output: $\lambda_\ell(\mathbf{X}_F)$

while *F has an internal edge e* **do**
  if *F has an independent edge e'* **then**
    $F \leftarrow F \setminus e'$
  else $F \leftarrow (F \odot e) \setminus \ell_e$;
let L be the list of orders of the connected components of F, in non-increasing order;
return λ;

Combinatorial Interpretation of coefficients

We have shown that various coefficients of the CSF of a tree in the star-basis are determined completely by properties of the tree. Below we record information about the relationship between the indexing partition λ and the value of c_λ in $\mathbf{X}_T$. Let $k = |I(T)|$.



- The coefficient on the partition $(n-m, 1^m)$ is given by:

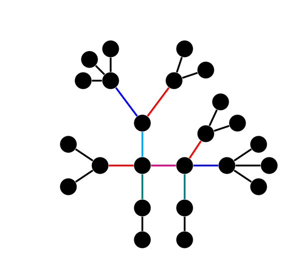
$$c_{(n-m,1^m)} = (-1)^m \binom{k}{m}$$

- For each $m = 2, \ldots, k+1$, we obtain:

$$\sum_{\ell(\lambda)=m} c_\lambda = 0$$

- From some of the results above, it follows that

$$\sum_{\substack{\ell(\mu)=2 \\ 1 \notin \mu}} c_\mu = k$$



- Lastly, if ω is the number of pairs of two adjacent edges that cannot be successively deleted in the DNC algorithm, we obtain:

$$\sum_{\ell(\lambda)=3} |c_\lambda| = 2\binom{k}{2} + k(k-1) - 2\omega$$

Acknowledgements

I would like to thank Professor Orellana for her mentorship and UGAR and the Kaminsky Fund for their financial support.

References

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2. Aliste-Prieto, J; De Mier, A; Orellana, R; Zamora, J; Marked Graphs and the Chromatic Symmetric Function, under review.
3. Gonzalez, M.; Koulogeroge, A.: Orellana, R.; Tomba, M.; The Chromatic Symmetric Function in the Star Basis, in progress.