The Chromatic Symmetric Function in the Star Basis
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Background and Definitions

- A graph $G = (V, E)$ is defined by a set of vertices $V = V(G)$ and edges $E = E(G)$.
- In a proper coloring of a graph, two adjacent vertices have different colors.
- The chromatic polynomial of a graph $\chi(G)$ gives the number of proper colorings of the graph $G$ with $k$ colors.
- The two graphs below are isomorphic:

\[
\begin{matrix}
\chi(G) = 12
\end{matrix}
\]

- An internal edge is an edge $uv$ such that $\deg(u), \deg(v) \geq 2$.
- The internal degree of a vertex $v$ is the number of internal edges incident to $v$.
- A leaf component of a forest $F$ is any maximal subtree $C \subseteq F$ such that every edge in $C$ is a leaf-edge in $F$.
- In 1994, Stanley defined the chromatic symmetric function (CSF) of a graph as a generalization of its chromatic polynomial:

\[
X_G = \sum_{\text{proper colorings } \lambda} x_1^{\lambda(1)} x_2^{\lambda(2)} \cdots
\]

where $x_1, x_2, \ldots$ are commuting variables.
- Stanley’s Tree Isomorphism Conjecture: If $T_1, T_2$ are non-isomorphic trees, then $x_{T_1} \neq x_{T_2}$

The CSF in the star basis

- A star graph on $n$ vertices $S_n(n)$ is a tree with $n - 1$ leaf vertices and one vertex of degree $n - 1$.

\[
\begin{matrix}
\chi_n = \sum_{k=0}^{n-1} \binom{n-1}{k} x_{1}^{n-1-k}
\end{matrix}
\]

- For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we define the star forest indexed by $\lambda$ as follows:

\[
S_{\lambda} = S_{\lambda_1} \cup \cdots \cup S_{\lambda_k}
\]

- For example, for $\lambda = (3, 3, 2)$,

\[
\begin{matrix}
\chi_{3,3,2} = \sum_{k=0}^{4} \binom{4}{k} x_{1}^{4-k}
\end{matrix}
\]

- Denote $X_{\lambda} = x_{\lambda}$. There is a well-known formula that expresses $X_{\lambda}$ in terms of the power-sum symmetric polynomials:

\[
X_{\lambda} = \sum_{k=0}^{\lambda} (-1)^{\lambda-k} \binom{n-1}{k} x_{1}^{n-1-k}
\]

- Similarly as above, for $\lambda = (\lambda_1, \ldots, \lambda_k)$, we have:

\[
X_{\lambda} = \sum_{k=0}^{\lambda} (-1)^{\lambda-k} \binom{n-1}{k} x_{1}^{n-1-k}
\]

- $(\lambda_1, \ldots, \lambda_k)$ is a basis for the algebra of symmetric functions of degree $n$.

Deletion-Near-Contraction (DNC)

- We can write the CSF compactly using the star-basis:

\[
X_G = \sum_{\lambda} x_{\lambda} \chi_{\rho}(\lambda)
\]

- The DNC relation expresses the CSF of a forest recursively as a linear combination of CSFs of three forests, each with fewer internal edges [2]:

\[
X_G = X_{G_{1}} - X_{G_{2}} - X_{G_{3}} + X_{G_{4}}
\]

- This relation allows us to compute the CSF efficiently.

- Once we fix a permutation of the internal edges of a tree, we can visualize the operations performed in the DNC algorithm using a ternary tree:

![DNC Diagram]

Diameter Results

- Let $(v_1, \ldots, v_n)$ be the set of internal vertices of a tree $T$ whose internal degree is strictly greater than one. Let $L_i$ be the set of leaf vertices adjacent to $v_i$.

The internal subgraph of $T$ is the subgraph $I \subseteq T$ induced by $(v_1, \ldots, v_n) \cup L_1 \cup \cdots \cup L_n$. The orders of the leaf components in $I$ are equal to $|L_1| + 1, \ldots, |L_n| + 1$.

- For a partition $\lambda$ of length $|T|$ containing no $1$’s, we define the edge-adjacency multiset in which each part $p$ of $\lambda$ appears with multiplicity $m_{\lambda}(p) = \max(\lambda_i, \lambda_j) - m_j(p)$:

\[
\begin{matrix}
\binom{1}{1} = 1, \binom{1}{2} = 1, \binom{2}{1} = 2, \binom{2}{2} = 3, \binom{3}{1} = 4, \binom{3}{2} = 4, \binom{3}{3} = 4
\end{matrix}
\]

- Theorem 1: If $T$ is any tree of diameter at most five, then the orders of the leaf components in the internal subgraph $I \subseteq T$ can be reconstructed from $X_T$.

- Corollary 1: If $T$ is a tree of diameter four, then $T$ can be reconstructed from $X_T$.

- Corollary 2: If $T$ is a tree of diameter five such that the orders of the leaf components in $I$ are distinct, then $T$ can be reconstructed from $X_T$.

- Theorem 2: If $T$ is a tree of diameter five such that the orders of the leaf components in $I$ are equal, then $T$ can be reconstructed from $X_T$ by extending edge-adjacency multisets to partitions of length $|I(T)| - 1$.

Leading Partition

A CSF has leading partition $\lambda$ if $c_0 = 0$ and $c_1 = 0$ for all $\mu$ with larger lexicographic ordering. Given any $\lambda \vdash n$, we say $\lambda$ is a hook partition of $n$ if $\lambda \geq 2$ and for some $1 \leq k \leq n - 1$,

\[
\begin{matrix}
\lambda = (k, k - 1, \ldots, 1)
\end{matrix}
\]

- Lemma 1: Given any non-hook partition of $n$, $\lambda$, Build-Leading-Tree returns a tree on $n$ vertices with leading partition $\lambda$.

- Algorithm 1: Get-Leading-Partition

Input: $\lambda \vdash n$

Output: $\lambda(x_T)$

while $F$ has an internal edge $e$ do

If $F$ has an independent edge $e$ then

$F' = (F - e) \setminus \{e\}$

let $L$ be the list of orders of the connected components of $F'$, in non-increasing order;

return $\lambda$;

Combinatorial Interpretation of coefficients

We have shown that various coefficients of the CSF of a tree in the star-basis are determined completely by properties of the tree. Below we record information about the relationship between the indexing partition $\lambda$ and the value of $c_k$ in $X_T$.

- Let $k = |I(T)|$.

- The coefficient on the partition $(m, m, 1^m)$ is given by:

\[
\binom{m}{m} = \binom{k}{m}
\]

- For each $m = 2, \ldots, k - 1$, we obtain:

\[
\sum_{c_k < k} c_k = 0
\]

- From some of the results above, it follows that

\[
\sum_{c_k > k} c_k = k
\]

- Lastly, if $c_k$ is the number of pairs of two adjacent edges that cannot be successively deleted in the DNC algorithm, we obtain:

\[
\sum_{c_k < k} c_k = \binom{k}{2} + k(k - 1) - 2\omega
\]

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References