Lattice Point Visibility Along Powers of Polynomials

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Abstract

A lattice point is **visible** from the origin when its coordinates are coprime — and precisely $6/\pi^2$ of all lattice points are visible. We study visibility along *polynomial* lines of sight: a lattice point is hidden along $F \in \mathbb{Z}[x]$ if an earlier point on the curve $y = tF(x)$ blocks it. We prove that for every $F = f^m$ with $\deg f \geq 2$ and $m \geq 2$, the *visibility density* equals 1 — the first infinite family for which this is known. When f is quadratic we count the rare invisible points to within a logarithm.

Background

Classical visibility.

A point $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ is **visible** from the origin if the open segment from $(0, 0)$ to (a, b) contains no other lattice point. Equivalently,

$$(a, b) \text{ visible} \iff \gcd(a, b) = 1,$$

so the density of visible points is the chance that two random integers are coprime:

$$\text{density} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \approx 0.6079.$$

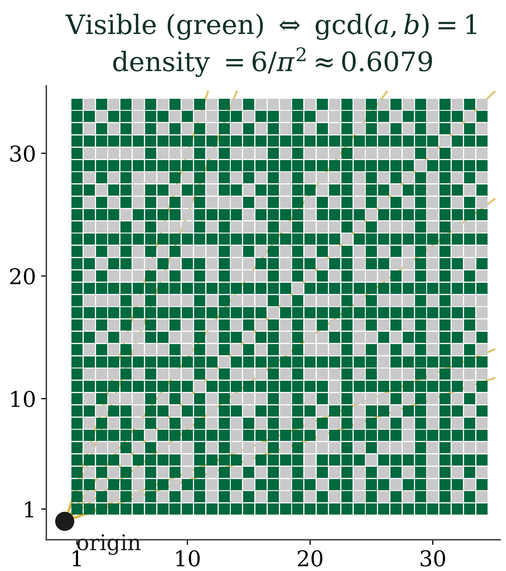

Figure 1: Visible (green) and blocked (gray) points; golden rays are lines of sight.

Polynomial lines of sight.

Replace the straight ray by the graph of a polynomial $F(x) \in \mathbb{Z}[x]$ with positive leading coefficient. A point $(a, h) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is **visible along** F if some rational t has $h = tF(a)$ with a the smallest positive integer u for which $tF(u)$ is a positive integer; otherwise it is **invisible**. Taking $F(x) = x$ recovers classical visibility. Everything changes the instant F acquires a *second* distinct root.

Main Results

The Visibility Density Conjecture.

The **visibility density** of F is

$$\mathrm{D}(F) = \lim_{N \to \infty} \frac{\#\{(a, b) \in [1, N]^2 : \text{ visible along } F\}}{N^2}.$$

Only two kinds of polynomials are known to keep a positive fraction of points invisible: nonzero constants and the single-root family $a(ux + v)^b$. Every other polynomial is conjectured to be "transparent": if $F \in \mathbb{Z}[x]$ has at least two distinct roots, then $\mathrm{D}(F) = 1$. Chaubey and Pandey [4] posed this conjecture for polynomials with $F(0) = 0$ and obtained a lower bound for certain quadratic families; we extend the setting to all of $\mathbb{Z}[x]$ and give the first density-one result.

Theorem (Lobsenz–Phillips).

Let $f(x) \in \mathbb{Z}[x]$ have degree ≥ 2, positive leading coefficient, and at least two distinct roots. Then for every integer $m \geq 2$ the polynomial $F = f(x)^m$ satisfies $\mathrm{D}(F) = 1$.

This is the first infinite family of polynomials for which the conjecture is verified. The proof is *effective*: it bounds the number of invisible points in $[1, N]^2$ by $O_{F,\varepsilon}\left(N^{1+1/\delta_F+\varepsilon}\right)$ for an integer $\delta_F \geq 2$ depending only on F.

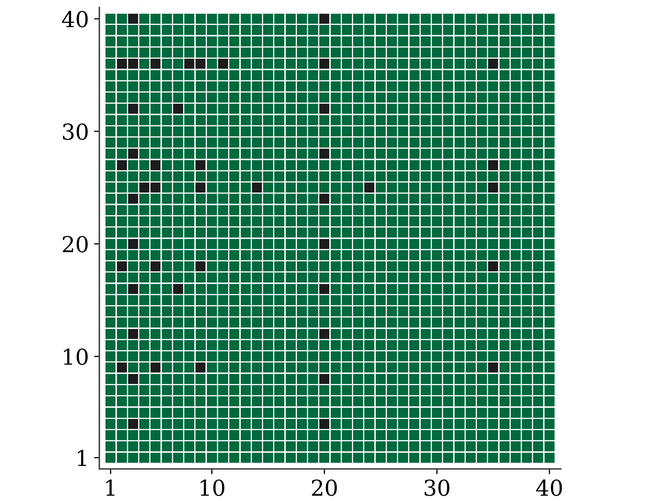

Figure 2: Visibility along $F(x) = (x^2 + x)^2$: the invisible points (black) thin to density zero.

Idea of the proof.

Counting invisible points reduces to a single sum built from the greatest common divisors of the values $F(a)$ and $F(b)$. The pairs that contribute are forced onto auxiliary *algebraic curves*, and Pila's theorem limits how many integer points such curves can carry. A Galois argument rules out the degenerate case; summing over all the curves shows the invisible set has density zero.

Sharper Bounds: The Quadratic Case

When f is quadratic, the auxiliary curves are conics.

Conics carry far fewer integer points than Pila's general bound allows, and this lets us sharpen the exponent $1 + 1/\delta_F$ all the way to 1, up to a logarithm.

Theorem (Lobsenz–Phillips, quadratic case).

Let $f(x) = Ax^2 + Bx + C$ with $A > 0$, $f(n) > 0$ for all $n \geq 1$, and nonzero discriminant; set $F = f^m$. Then

- $m \geq 3$: $\#\mathrm{Invisible}_F(N) \ll_F N \log N$;
- $m = 2$: $\#\mathrm{Invisible}_F(N) \ll_{F,\varepsilon} N^{1+\varepsilon}$.

Conics and Pell equations.

Completing the square turns each auxiliary conic into a generalized Pell equation $X^2 - DY^2 = n$. Its integer solutions are rigid: they fall into only finitely many families, and within each family the solutions grow geometrically. So a window of size N holds only $O(\log N)$ of them — the source of the logarithmic savings.

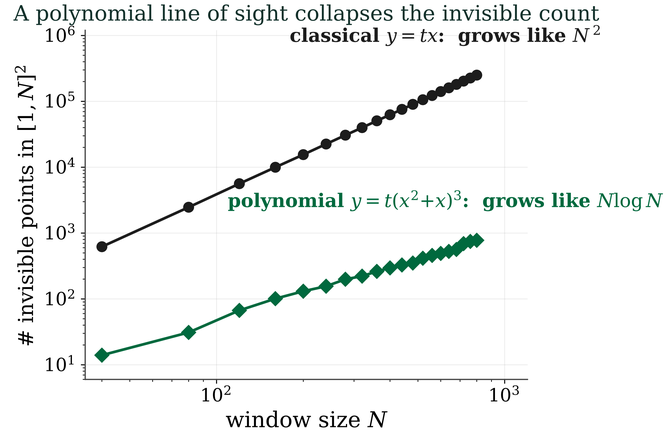

Figure 3: A polynomial line of sight collapses the invisible count from order N^2 (classical) to order N.

Outlook.

The full Visibility Density Conjecture — arbitrary polynomials with two distinct roots — remains open.

References

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[4] S. Chaubey, A. K. Pandey. *On the density of visible lattice points along polynomials*, arXiv:2109.08431, 2021.

[5] J. Pila. *Density of integer points on plane algebraic curves*. Int. Math. Res. Not. **18** (1996).