Multiplicative Problems in Combinatorial Number Theory

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Abstract

In this thesis we look at several problems that lie in the intersection between combinatorial and multiplicative number theory. A common theme of many of these problems are estimates for and properties of the smooth numbers, those integers not divisible by any large prime factors.

The first is the Ramsey-theoretic problem to determine the maximal size of a subset of the integers containing no 3-term geometric progressions. This problem was first considered by Rankin, who constructed such a subset with density about 0.719. By considering progressions among the smooth numbers, we demonstrate a method to effectively compute the greatest possible upper density of a set of integers avoiding geometric progressions.

Second, we consider the problem of determining which prime number occurs most frequently as the largest prime divisor on the interval $[2, x]$, as well as the set prime numbers which ever have this property for some value of x, a problem closely related to the analysis of factoring algorithms. This set of primes appears to be related to another subset of primes, those primes which form the vertex points of the convex hull of the prime number graph, which we consider next, obtaining improved bounds for the count of these primes up to x.

The largest prime divisor also plays a role in a more probabilistic problem we consider next, a random multiplicative walk on the residues modulo n. Finally, we consider the k-Lehmer numbers, a subset of composite integers closely related to the Carmichael numbers, and improve existing estimates for the distribution of these integers.