A function-field analogue of Conway's topograph

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In [1], Conway introduces a new visual method to display values represented by an integral binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2 \in \mathbb{Z}[x, y]$. This topograph method, as he calls it, leads to a simple and elegant method of classifying all integral binary quadratic forms and answering some basic questions about them. In particular, Conway shows that the topograph of any definite binary quadratic form has a unique "well", while the topograph of any indefinite binary quadratic form has a unique "river". In our work, we develop an analogue of Conway's topograph method in the function-field setting, that is, for binary quadratic forms with coefficients in $\mathbb{F}_q[T]$, where q is an odd prime power.

Our starting point is the connection between Conway's topograph method and hyperbolic geometry. Conway himself notes that it is most natural to consider the 3-regular tree central to his approach as embedded on the hyperbolic plane. To set the notation, let $A = \mathbb{F}_q[T]$ be the ring of polynomials over a finite field of odd order q and $K = \mathbb{F}_q(T)$ its field of fractions. The completion of K with respect to the usual valuation arising from the degree function is $\widehat{K} = \mathbb{F}_q((T^{-1}))$. In [2], Paulin provides an interpretation of continued fraction expansions of elements of $\widehat{K}$ in terms of the action of $\mathrm{SL}_2(A)$ on the Bruhat–Tits tree $\mathcal{T}_{q+1}$ of $\mathrm{SL}_2(\widehat{K})$. Since the classical reduction theory for indefinite integral binary quadratic forms relies on the theory of continued fractions, Paulin's work led us to consider $\mathcal{T}_{q+1}$ as a suitable function-field analogue of the hyperbolic plane.

After we recast the underlying infrastructure of Conway's topograph in terms of constructions on $\mathcal{T}_{q+1}$, we formulate and prove an analogue of Conway's climbing lemma. We then show that just as in the classical setting, there is a unique “well” (respectively, “river”) on the topograph of any definite (respectively, “indefinite”) binary quadratic form over A.

References

[1] John H. Conway, *The sensual (quadratic) form*, Carus Mathematical Monographs, vol. 26, Mathematical Association of America, Washington, DC, 1997, With the assistance of Francis Y. C. Fung. MR 1478672 (98k:11035)

[2] Frédéric Paulin, *Groupe modulaire, fractions continues et approximation diophantienne en caractéristique* p, Geom. Dedicata **95** (2002), 65–85. MR 1950885 (2003j:20043)