Dynamics of Magnetic Flows on Nilmanifolds

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Abstract

In this thesis, we study the dynamics of magnetic flows on compact nilmanifolds. Magnetic flows are generalizations of geodesic flows. They model the motion of a particle of unit mass and unit charge in a smooth manifold \( M \) in the presence of a magnetic field. As such, their dynamical properties are influenced by both the underlying Riemannian geometry and the closed 2-form on \( M \) which plays the role of the magnetic field. At the same time, nilmanifolds are a rich and varied class of examples, as well as a source of conjectures and counterexamples in Riemannian geometry. More precisely, we consider nilmanifolds of the form \( M = \Gamma \backslash G \), where \( G \) is a simply connected 2-step nilpotent Lie group and \( \Gamma < G \) is a cocompact discrete subgroup. The manifold \( M \) is endowed with a Riemannian metric \( g \) and closed 2-form, or magnetic field, \( \sigma \), each of which pulls back to a left-invariant tensor field on \( G \).

First, we focus on the case when \( G \) is the \( 2n + 1 \) dimensional Heisenberg group, and \( \sigma \) is exact. We calculate the Mañé critical value and the lengths of closed magnetic geodesics in nontrivial free homotopy classes. Next we consider the topological entropy of magnetic flows on arbitrary 2-step compact nilmanifolds. When \( \sigma \) represents a rational cohomology class and its restriction to \( g = T_eG \) vanishes on the derived algebra, we prove that the associated magnetic flow has zero topological entropy on a dense set of energy levels. In particular, this is the case when \( \sigma \) represents a rational cohomology class and is exact. Lastly, we provide an example of a magnetic field on a 2-step compact nilmanifold that has positive topological entropy for arbitrarily high energy levels. The salient difference in this case is that \( \sigma \) is not exact. We discuss the relationship to Mañé’s critical value. The main tool is a symplectic reduction of the cotangent bundle of a nilmanifold of one more dimension than \( M \).