Combinatorial proofs of linear algebraic identities

Melanie Dennis

Abstract

In this thesis, we examine two determinantal identities: the Lewis Carroll Identity and the Dodgson/Muir Identity. The Lewis Carroll Identity expresses the determinant of a matrix in terms of subdeterminants obtained by deleting one row and column or a pair of rows and columns. Using the Matrix Tree Theorem, we convert this into an equivalent identity involving sums over pairs of forests. Unlike the Lewis Carroll Identity, the Forest Identity involves no minus signs. Using the Involution Principle, we can pull back Zeilberger's proof of the Lewis Carroll Identity to a bijective proof of the Forest Identity. This bijection is implemented by the Red Hot Potato algorithm, so called because the way edges get tossed back and forth between the two forests is reminiscent of the children's game of hot potato. We examine in detail the connection between the Red Hot Potato algorithm and Zeilberger's proof. The Dodgson/Muir Identity is a generalization of the Lewis Carroll Identity that expresses the determinant of a matrix in terms of subdeterminants obtained by deleting sets of $k-1$ rows and columns or sets of k rows and columns. Again, we find the equivalent identity involving sums over k-tuples of forests. We give a direct proof of the Dodgson/Muir Identity using pairwise iterations of the Red Hot Potato algorithm.