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The Reciprocal Sum of the Amicable Numbers

A thesis submitted in partial fulfillment
of the requirements for the degree

Bachelor of Arts

in

Mathematics

by

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2014

Abstract

Two different positive integers a and b are said to form an *amicable* pair if $s(a) = b$ and $s(b) = a$, where $s(n)$ denotes the sum of proper divisors of n. In 1981, Pomerance proved that the count of amicable numbers up to x is less than $x/\exp\left((\log x)^{1/3}\right)$ for x large. It follows immediately that the sum P of the reciprocals of numbers belonging to an amicable pair is a constant. In 2011, Bayless & Klyve showed that $P < 656{,}000{,}000$. In this paper, we improve this upper bound by proving that $P < 4084$, based on recent work by Pomerance that shows a stricter bound for the count of amicable numbers plus some other ideas that are new to this thesis.

Acknowledgements

First and foremost, I would like to thank my thesis advisor, Prof. Carl Pomerance, for his continued patience and guidance throughout this project. I also want to express my deepest gratitude to my professors at the Department of Mathematics for their support and mentorship during the past four years.

Contents

1 Introduction

Let σ denote the sum-of-divisors function, and let $s(n) = \sigma(n) - n$ denote the sum-of-proper-divisors function. Two different positive integers n and n' are said to form an amicable pair if and only if $s(n) = n'$ and $s(n') = n$. We say n is amicable if it belongs to an amicable pair.

The smallest pair of amicable numbers, 220 and 284, was first known to Pythagoras in the 5^{th} century BC, and was ascribed with many mystical properties. In the 9^{th} century, Thâbit ibn Qurra found a formula for amicable pairs, which states that if $p = 3 \cdot 2^n - 1$, $q = 3 \cdot 2^{n-1} - 1$, $r = 9 \cdot 2^{2n-1} - 1$ are primes, then $2^n pq$, $2^n r$ form an amicable pair. For example, $n = 2$ yields the primes $p = 11, q = 5, r = 71$, which generates the pair $(220, 284)$.

It was not until 1636 that a second pair of amicable numbers $(17296, 18416)$, which corresponds to $n = 4$, was found by Fermat. Later, Descartes gave a third pair of amicable numbers (9363584, 9437056), which corresponds to $n = 7$.

In the 18^{th} century, Euler generalized Thâbit's rule: if $p = (2^{n-m} + 1)2^n - 1$, $q = (2^{n-m} + 1) \cdot 2^{n-1} - 1$, $r = (2^{n-m} + 1)^2 \cdot 2^{2n-1} - 1$ are primes, then $2^n pq$, $2^n r$ form an amicable pair. He also compiled a list of 64 amicable pairs, of which two were later shown to be not amicable. Interestingly, Euler completely overlooked the second smallest amicable pair $(1184, 1210)$, which was discovered by Paganini, a 16-year-old Italian, in 1866.

We now know exhaustively all amicable pairs with the smallest member up to 10^{14} and nearly 12 million other pairs [6]. However, the infinitude of the set of amicable numbers has yet to be proved.

Let $\mathcal{A}$ denote the set of amicable numbers, and let $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$. In 1954, Kanold initiated the quest to explore the asymptotic density of $\mathcal{A}$, showing that $\#\mathcal{A}(x) < 0.204x$ for x sufficiently large [4]. In 1955, Erdős was the first to prove that the set of amicable numbers has zero asymptotic density, showing that $\#\mathcal{A}(x) = o(x)$ [2]. From 1973 to 1981, Rieger, Erdős, and Pomerance improved upon this result, proving that $\#\mathcal{A}(x)$ is less than $x/(\log\log\log\log x)^{1/2}$, $O\left(x/\log\log\log x\right)$, $x/\exp\left((\log\log\log x)^{1/2}\right)$, and $x/\exp\left((\log x)^{1/3}\right)$ for x sufficiently large [10][3][7][8]. Most recently in 2014, Pomerance showed that $\#\mathcal{A}(x) \leq x/\exp((\log x)^{1/2})$ as $x \to \infty$ [9].

It follows immediately from the last two results that the reciprocal sum of the amicable numbers is finite. In 2011, Bayless & Klyve showed a numerical bound for the reciprocal sum of amicable numbers, which they denote by P:

$$0.0119841556 \leq P < 656{,}000{,}000.$$

In this paper, we improve Bayless & Klyve's result in [1] by showing a smaller upper bound for the reciprocal sum of amicable numbers.

Theorem 1.1. *We have*

$$P = \sum_{n \in \mathcal{A}} \frac{1}{n} < 4084.$$

We will introduce some standard notations and describe the framework in part 2, prove some useful lemmas in part 3, and give detailed proof and numerical estimates in part 4.

2 Framework Description

Let $n \in \mathcal{A}$ be an amicable number and $n' = s(n)$ denote the corresponding friend. For some $K > \log 10^{14}$, we can write

$$P = \sum_{n\in\mathcal{A}} \frac{1}{n} = \sum_{\substack{n\in\mathcal{A}\\ \min\{n,n'\}\le 10^{14}}} \frac{1}{n} + \sum_{\substack{n\in\mathcal{A}\\ 10^{14}<\min\{n,n'\}\le \exp(K)}} \frac{1}{n} + \sum_{\substack{n\in\mathcal{A}\\ \min\{n,n'\}>\exp(K)}} \frac{1}{n}.$$

From the exhaustive list of amicable pairs with the smaller member less than 10^{14}, we can compute

$$P_s = \sum_{\substack{n\in\mathcal{A}\\ \min\{n,n'\}\le 10^{14}}} \frac{1}{n} = 0.011984156739048\ldots$$

The list of amicable numbers is obtained from [6], and the reciprocal sum is computed in MatLab with standard machine precision. This result differs from the lower bound shown in [1] by approximately 10^{-9}.

An amicable pair (n, n') can fall into one of the following cases: (n, n') are both odd, (n, n') are both even, or (n, n') are of different parity.

If n and n' are of different parity, then $\sigma(n) = \sigma(n') = n + n'$ is odd. Since $\sigma(p^a) = 1 + p + p^2 + \cdots + p^a$ is odd if and only if $p = 2$ or $2|a$, the odd member is a square, and the even member is either a square or twice one. No odd-even amicable pair has been found, so the reciprocal sum of such amicable numbers, if they exist at all, is small and bounded by

$$\varepsilon < \frac{3}{2} \sum_{a>10^{14}} \frac{1}{a^2} < 1.5 \times 10^{-14}.$$

From this point forward, we consider only amicable pairs whose members are either both odd or both even.

2.1 Small amicable numbers

For amicable pairs (n, n') such that $10^{14} < \min\{n, n'\} \le \exp(K)$, we will grossly overestimate their contributions to P since the asymptotic behavior outlined by the proof has yet to kick in. In particular, we will consider even amicable pairs and the odd ones separately.

If n and n' are both even, then

$$\frac{n'}{n} = \sum_{\substack{d|n\\ d>1}} \frac{1}{d} > \frac{1}{2},$$

and similarly $n/n' > 1/2$, so $10^{14} < n, n' \le 2\exp(K)$.

If n and n' are odd, then $\min\{n, n'\}$ is an odd abundant number. Recall that an integer is abundant if the sum of its proper divisors is greater than the number itself.

We have

$$P(K) = \sum_{\substack{n\in\mathcal{A}\\ 10^{14}<\min\{n,n'\}\le\exp(K)}} \frac{1}{n} = \sum_{\substack{n\in\mathcal{A}\\ n \text{ even}\\ 10^{14}<n\le 2\exp(K)}} \frac{1}{n} + \sum_{\substack{n\in\mathcal{A}\\ n \text{ odd}\\ 10^{14}<\min\{n,n'\}\le\exp(K)}} \frac{1}{n}$$

$$\le \frac{1}{2}\left(\sum_{10^{14}/2<n\le\exp(K)} \frac{1}{n}\right) + 2\left(\sum_{\substack{n \text{ odd abundant}\\ 10^{14}<n\le\exp(K)}} \frac{1}{n}\right),$$

where

$$\sum_{10^{14}/2<n\le\exp(K)} \frac{1}{n} < \int_{10^{14}/2}^{\exp(K)} \frac{dt}{t} = K - \log(10^{14}/2),$$

and

$$\sum_{\substack{n \text{ odd abundant}\\ 10^{14}<n\le\exp(K)}} \frac{1}{n}$$

is smaller. Let's find out how small it is.

Let $h(n) = \sigma(n)/n$, then h is a multiplicative function. Let f_j be the multiplicative function defined as follow

$$\begin{cases} f(1) = 1 \\ f_j(p^a) = h(p^a)^j - h(p^{a-1})^j \end{cases}.$$

For $x \ge 3$, we have

$$\sum_{\substack{n\le x\\ n \text{ odd}}} h(n)^j = \sum_{\substack{n\le x\\ n \text{ odd}}} \sum_{d|n} f_j(d),$$

so that

$$\sum_{\substack{n\le x\\ n \text{ odd}}} \frac{h(n)^j}{n} = \sum_{\substack{n\le x\\ n \text{ odd}}} \frac{1}{n}\sum_{d|n} f_j(d) = \sum_{\substack{d\le x\\ d \text{ odd}}} f_j(d) \sum_{\substack{1\le m\le x/d\\ m \text{ odd}}} \frac{1}{md} \le \sum_{\substack{d\le x\\ d \text{ odd}}} \frac{f_j(d)}{d} \sum_{\substack{1\le m\le x\\ m \text{ odd}}} \frac{1}{m}.$$

Here,

$$\sum_{\substack{1\le m\le x\\ m \text{ odd}}} \frac{1}{m} \le 1 + \sum_{k=1}^{\lfloor x/2\rfloor} \frac{1}{2k+1} < 1 + \frac{1}{2}\sum_{k=1}^{\lfloor x/2\rfloor} \frac{1}{k} < 1 + \frac{1+\log(x/2)}{2} < \frac{\log x + 2.307}{2},$$

and

$$\sum_{\substack{d\le x\\ d \text{ odd}}} \frac{f_j(d)}{d} \le \sum_{d \text{ odd}} \frac{f_j(d)}{d} = \prod_{p \text{ odd}} \left(1 + \frac{f_j(p)}{p} + \frac{f_j(p^2)}{p^2} + \dots\right) = \pi_j, \text{say},$$

is finite, as we shall see in Lemma 3.7.

Since $h(n) > 2$ for all n abundant, we have

$$\sum_{\substack{n \le x \\ n \text{ odd abundant}}} \frac{1}{n} \le \frac{1}{2^j} \sum_{\substack{n \le x \\ n \text{ odd}}} \frac{h(n)^j}{n} \le \frac{\pi_j(\log x + 2.307)}{2^{j+1}}.$$

After investigating $j \in [1, 40]$, we pick $j = 18$, which yields $\pi_j < 6231.87$, so that

$$\sum_{\substack{n \le x \\ n \text{ odd abundant}}} \frac{1}{n} \le 0.01189 \log x + 0.02743.$$

Thus, we have

$$P(K) \le 0.52378K - 15.71668.$$

2.2 Large amicable numbers

For the remaining pairs, i.e., (n, n') such that $\min\{n, n'\} > \exp(K)$, we will consider them in small intervals and build an *ordered* list of properties.

Let $\mathcal{A}_k = \{n \in \mathcal{A} : e^{k-1} < n < e^k\}$, and let $\omega(n)$ denote the number of distinct primes dividing n.

Property (1): For $n \in \mathcal{A}_k$, $\omega(n) \le \lfloor 4 \log k \rfloor$.

Let $\mu_k = \prod_{i=2}^{\lceil 4 \log k \rceil} \frac{p_i}{p_i - 1}$, where p_i is the i^{th} prime. We will show that for $n \in \mathcal{A}_k$ such that n and n' have property (1), $n/\mu_k \le n' \le n\mu_k$.

Let $L_k = \exp(\sqrt{k}/5)$. Recall that a positive integer s is squarefull if and only if for every prime p dividing s, p^2 also divides s. Also recall that a positive integer is squarefree if and only if it is not divisible by any perfect square. Let $P(\cdot)$ denote the largest-prime-divisor function.

Property (2): For $n \in \mathcal{A}_k$, the largest squarefull divisor of n is at most $L_k/4$.

Property (3): For $n \in \mathcal{A}_k$, the largest squarefree divisor d of n with $P(d) \le L_k$ satisfies $d \le e^{k/3}$.

Property (4): For $n \in \mathcal{A}_k$, $P(\gcd(n, s(n))) \le L_k/2$.

Let $n = pm$, $n' = p'm'$ where p and p' are the largest prime factors of n and n'. Note that properties (2), (3), and (4) imply that $p \ne p', p \nmid m$, and $p' \nmid m'$.

Property (5): For $n \in \mathcal{A}_k$, $mm' \ge \frac{e^k}{L_k}$.

Property (6): For $n \in \mathcal{A}_k$, $p \le e^{3k/4} L_k$.

Write $m = m_0 m_1$ where m_0 is the largest divisor of m such that m_0 is either a squarefull number or twice one, then m_1 is odd, squarefree, and coprime to m_0. Let $q = P(m_1)$, and write $q + 1 = q_0 q_1$ where q_0 is the largest divisor of $q + 1$ such that q_0 is either a squarefull number or twice one.

Property (7): For $n \in \mathcal{A}_k$, q_0 is at most $16L_k$.

Property (8): For $n \in \mathcal{A}_k$, $P(\sigma(m_1)) \geq L_k$.

For each property (i), let

$$P^{(\mathbf{i})} = \sum_{\substack{n\in\mathcal{A} \\ \min\{n,n'\}>\exp(K) \\ n,n' \text{ pass } (\mathbf{1})-(\mathbf{i\text{-}1}) \\ n \text{ or } n' \text{ fails } (\mathbf{i})}} \frac{1}{n}.$$

Then,

$$P^{(\mathbf{i})} \leq \sum_{k=K+1}^{\infty} \sum_{\substack{n\in\mathcal{A}_k \\ n,n' \text{ pass } (\mathbf{1})-(\mathbf{i\text{-}1}) \\ n \text{ fails } (\mathbf{i})}} \left(\frac{1}{n} + \frac{1}{n'}\right) = \sum_{k=K+1}^{\infty} s_k^{(\mathbf{i})}, \text{ say.}$$

Finally, we are left with $n \in \mathcal{A}_k$ where both n and n' have properties (1)-(8). We proceed to estimate

$$P^{(*)} = \sum_{\substack{n\in\mathcal{A} \\ \min\{n,n'\}>\exp(K) \\ n,n' \text{ pass } (\mathbf{1})-(\mathbf{8})}} \frac{1}{n} = \sum_{k=K+1}^{\infty} \sum_{\substack{n\in\mathcal{A}_k \\ P(n)>P(n') \\ n,n' \text{ passes } (\mathbf{1})-(\mathbf{8})}} \left(\frac{1}{n} + \frac{1}{n'}\right) = \sum_{k=K+1}^{\infty} s_k^{(*)}, \text{ say,}$$

so that

$$\sum_{\substack{n\in\mathcal{A} \\ \min\{n,n'\}>\exp(K)}} \frac{1}{n} \leq P^{(*)} + \sum_{\mathbf{i}=1}^{8} P^{(\mathbf{i})}.$$

To optimize the sum of the expression above and $P(K)$, we choose $K = 5935$.

In this paper, we will repeatedly use ϵ, ι, and c to denote different constants. In all cases, the explicit formula for these constants are clear from the context and are used in computation. The naming of constants follows the convention that $\epsilon \to 0$ and $\iota \to 1$ as $k \to \infty$. The letter p, q, and r are reserved for prime variables. We use ζ and φ to denote the Riemann zeta function and the Euler phi function, respectively. We also use $\pi(x; d, a)$ to denote the number of primes p congruent to a modulo d with $p \leq x$.

3 Lemmas

Lemma 3.1. *For all* $k, d, v \in \mathbb{Z}^+, z > 0$,

$$\sum_{z \le v \le ez} \frac{1}{v} \le \min\left\{1 + \frac{1}{z}, \frac{3}{2}\right\}.$$

Proof. If $z \le 1$, then

$$\sum_{z \le v \le ez} \frac{1}{v} \le \sum_{1 \le v \le e} \frac{1}{v} = \frac{3}{2}.$$

If $1 < z \le 2$, then

$$\sum_{z \le v \le ez} \frac{1}{v} \le \sum_{1 < v \le 2e} \frac{1}{v} = 1.28\bar{3} < \frac{3}{2}.$$

If $z > 2$, then

$$\sum_{z < v < ez} \frac{1}{v} \le \frac{1}{z} + \int_z^{ez} \frac{1}{v} dv = 1 + \frac{1}{z}.$$

□

Lemma 3.2. *For all* $x > 286$, *we have*

$$\mathcal{H}(x) := \sum_{p \le x} \frac{1}{p} \le \log\log x + B + \frac{1}{2\log^2 x} \tag{3.1}$$

and

$$\mathcal{H}_1(x) := \sum_{p \le x} \frac{1}{p-1} \le \log\log x + B + \frac{1}{2\log^2 x} + D, \tag{3.2}$$

where $B = 0.261497\ldots$ *and* $D = \sum_p 1/p(p-1) = 0.773157\ldots$.

Proof. The inequality (3.1) is from [11], and (3.2) follows immediately. □

Lemma 3.3. *: For all* $z > 286$,

$$\sum_{z < p \le ez} \frac{1}{p} \le \frac{1}{\log z} + \frac{1}{\log^2 z}.$$

Proof. From equations (3.17) and (3.18) of [11], we have

$$\begin{aligned}\sum_{z < p \le ez} \frac{1}{p} &= \sum_{p \le ez} \frac{1}{p} - \sum_{p \le z} \frac{1}{p} \le \log(1 + \log z) + B + \frac{1}{2(1+\log z)^2} - \log\log z - B + \frac{1}{2\log^2 z} \\ &= \log\left(1 + \frac{1}{\log z}\right) + \frac{1}{2(1+\log z)^2} + \frac{1}{2\log^2 z} \le \frac{1}{\log z} + \frac{1}{\log^2 z}.\end{aligned}$$

□

Lemma 3.4. *: For all $w \in \mathbb{Z}^+, x \geq 0$ such that $\mathcal{H}(x) < w+1$, we have*

$$\sum_{\substack{\omega(d)\geq w \\ P(d)\leq x \\ d \text{ squarefree}}} \frac{1}{d} \leq \frac{\mathcal{H}(x)^w}{w!} \cdot \frac{w+1}{w+1-\mathcal{H}(x)}.$$

Proof. By the multinomial theorem, we have

$$\begin{aligned}
\sum_{\substack{\omega(d)\geq w \\ P(d)\leq x \\ d \text{ squarefree}}} \frac{1}{d} &\leq \sum_{j=w}^{\infty} \frac{1}{j!} \left(\sum_{p\leq x} 1/p \right)^j \\
&= \frac{\mathcal{H}(x)^w}{w!} \left(1 + \frac{\mathcal{H}(x)}{w+1} + \frac{\mathcal{H}(x)^2}{(w+1)(w+2)} + \dots \right) \\
&\leq \frac{\mathcal{H}(x)^w}{w!} \left(1 + \frac{\mathcal{H}(x)}{w+1} + \frac{\mathcal{H}(x)^2}{(w+1)^2} + \dots \right) \\
&= \frac{\mathcal{H}(x)^w}{w!} \cdot \frac{w+1}{w+1-\mathcal{H}(x)}.
\end{aligned}$$

□

Lemma 3.5. *For all $d \in \mathbb{Z}^+, d, x \geq 3$,*

$$\sum_{\substack{p\equiv -1 \pmod d \\ d<p\leq x}} \frac{1}{p} \leq \frac{2(\log\log x - \log\log(2-1/d) + 1/\log(x/d))}{\varphi(d)}.$$

Proof. From the Brun-Titchmarsh theorem [5], we have

$$\pi(x; d, -1) \leq \frac{2x}{\varphi(d)\log(x/d)}.$$

Using partial summation, we have

$$\begin{aligned}
\sum_{\substack{p\equiv -1 \pmod d \\ d<p\leq x}} \frac{1}{p} &\leq \frac{2}{\varphi(d)\log(x/d)} + \frac{2}{\varphi(d)} \int_{2d-1}^{x} \frac{dt}{t\log(t/d)} \\
&\leq \frac{2}{\varphi(d)} \left(\frac{1}{\log(x/d)} + \log\log(t/d)\Big|_{2d-1}^{x} \right) \\
&= \frac{2}{\varphi(d)} \left(\frac{1}{\log(x/d)} + \log\log(x/d) - \log\log(2-1/d) \right).
\end{aligned}$$

□

Lemma 3.6. *For $x \geq 1$,*

$$\mathcal{S}_0(x) := \sum_{\substack{s \leq x \\ s \text{ squarefull}}} 1 \leq \zeta(3/2)\sqrt{x},$$

$$\mathcal{S}_1(x) := \sum_{\substack{s \geq x \\ s \text{ squarefull}}} \frac{1}{s} \leq \frac{3}{\sqrt{x}}.$$

For $x \geq 2$,

$$\mathcal{S}_*(x) := \sum_{\substack{s \geq x \\ s \text{ squarefull}}} \frac{\log\log s}{s} \leq \frac{3\log\log x + 3\log 2}{\sqrt{x}} + 2\zeta(3/2)\left(\frac{\log\log x + \log 2}{x} + \frac{1}{2x\log x}\right).$$

Proof. Since every squarefull number can be written as a product of a square and a cube,

$$\mathcal{S}_0(x) \leq \sum_{a^2b^3 \leq x} 1 = \sum_{b \leq x^{1/3}} \sum_{a \leq (x/b^3)^{1/2}} 1 \leq \sum_{b \leq x^{1/3}} \left(\frac{x}{b^3}\right)^{1/2} = \sqrt{x} \sum_{b \leq x^{1/3}} \frac{1}{b^{3/2}} \leq \zeta(3/2)\sqrt{x}.$$

We can verify $\mathcal{S}_1(x) \leq 3/\sqrt{x}$ for $x \leq 14000$ using $\mathcal{S}_1(1) = \zeta(2)\zeta(3)/\zeta(6)$. For $x > 14000$, we have

$$\begin{aligned}
\mathcal{S}_1(x) &\leq \sum_{a^2b^3 \geq x} \frac{1}{a^2b^3} \\
&\leq \sum_{b \leq x^{1/3}} \frac{1}{b^3} \sum_{a \geq (x/b^3)^{1/2}} \frac{1}{a^2} + \sum_{b \geq x^{1/3}} \frac{1}{b^3} \sum_{a} \frac{1}{a^2} \\
&\leq \sum_{b \leq x^{1/3}} \frac{1}{b^3}\left(\frac{b^3}{x} + \left(\frac{b^3}{x}\right)^{1/2}\right) + \zeta(2) \sum_{b \geq x^{1/3}} \frac{1}{b^3} \\
&\leq \frac{1}{x^{2/3}} + \frac{1}{x^{1/2}} \sum_{b \leq x^{1/3}} \frac{1}{b^{3/2}} + \zeta(2)\left(\frac{1}{x} + \frac{1}{2x^{2/3}}\right) \\
&\leq \frac{\zeta(3/2)}{x^{1/2}} + \frac{1 + \zeta(2)/2}{x^{2/3}} + \frac{\zeta(2)}{x} \\
&\leq \frac{3}{\sqrt{x}}.
\end{aligned}$$

By partial summation and our inequality for $\mathcal{S}_0$, we also have

$$\begin{aligned}
\mathcal{S}_*(x) &= \lim_{z\to\infty} \sum_{\substack{x \leq s \leq z \\ s \text{ squarefull}}} \frac{\log\log s}{s} \\
&= \lim_{z\to\infty} \left(\frac{(\mathcal{S}_0(z) - \mathcal{S}_0(x))\log\log z}{z} - \int_x^z (\mathcal{S}_0(t) - \mathcal{S}_0(x))\, d\left(\frac{\log\log t}{t}\right)\right) \\
&= \int_x^\infty \mathcal{S}_0(t)\left(\frac{\log\log t}{t^2} - \frac{1}{t^2\log t}\right) dt
\end{aligned}$$

$$\leq \zeta(3/2)\int_x^\infty \left(\frac{\log\log t}{t^{3/2}} - \frac{1}{t^{3/2}\log t}\right) dt$$
$$= \zeta(3/2)\int_x^\infty \left(\frac{\log\log t}{t^{3/2}} - \frac{2}{t^{3/2}\log t} + \frac{1}{t^{3/2}\log t}\right) dt$$
$$\leq \zeta(3/2)\int_x^\infty \left(\frac{\log\log t}{t^{3/2}} - \frac{2}{t^{3/2}\log t} + \frac{1}{t^{3/2}\log x}\right) dt$$
$$= \zeta(3/2)\left(-\frac{2\log\log t}{\sqrt{t}} - \frac{2}{\sqrt{t}\log x}\Bigg|_x^\infty\right)$$
$$= 2\zeta(3/2)\left(\frac{\log\log x}{\sqrt{x}} + \frac{1}{\sqrt{x}\log x}\right).$$

Now, we can improve this estimate:

$$\mathcal{S}_*(x) = \sum_{\substack{x\leq s<x^2 \\ s \text{ squarefull}}} \frac{\log\log s}{s} + \sum_{\substack{s\geq x^2 \\ s \text{ squarefull}}} \frac{\log\log s}{s}$$
$$\leq \log\log x^2 \sum_{\substack{s\geq x \\ s \text{ squarefull}}} \frac{1}{s} + 2\zeta(3/2)\left(\frac{\log\log x + \log 2}{x} + \frac{1}{2x\log x}\right)$$
$$\leq \frac{3\log\log x + 3\log 2}{\sqrt{x}} + 2\zeta(3/2)\left(\frac{\log\log x + \log 2}{x} + \frac{1}{2x\log x}\right).$$

□

Lemma 3.7. *Let f be the multiplicative function defined as in 2.1. For all $j \in \mathbb{Z}^+$,*

$$\pi_j = \prod_{p \text{ odd}} \left(1 + \frac{f_j(p)}{p} + \frac{f_j(p^2)}{p^2} + \dots\right)$$

is a finite number.

Proof. First, we have

$$\frac{f_j(p)}{p} = \frac{h(p)^j - 1}{p} = \frac{\left(1+\frac{1}{p}\right)^j - 1}{p} = O\left(\frac{1}{p^2}\right).$$

Because $0 < f_j(p^a) < h(p^a)^j$, for all $a \geq 2$,

$$\frac{f_j(p^a)}{p^a} \leq \frac{h(p^a)^j}{p^a} \leq \frac{\sigma(p^a)^j}{p^{a+ja}} \leq \left(\frac{p}{p-1}\right)^j \frac{1}{p^a}.$$

Therefore, for each fixed $j \in \mathbb{Z}^+$

$$1 + \frac{f_j(p)}{p} + \frac{f_j(p^2)}{p^2} + \dots = 1 + O\left(\frac{1}{p^2}\right),$$

so that

$$\log \pi_j = O\left(\sum_{p \text{ odd}} \frac{1}{p^2}\right) = O(1).$$

□

In order to estimate an upper bound for π_j, we choose large integers A, B, and compute

$$\prod_{p \le B}\left(1 + \frac{f_j(p)}{p} + \frac{f_j(p^2)}{p^2} + \dots\right) \le \prod_{p \le B}\left(1 + \sum_{a=1}^{A} \frac{f_j(p^a)}{p^a} + \left(\frac{p}{p-1}\right)^{j+1} \frac{1}{p^{A+1}}\right),$$

and

$$\prod_{p > B}\left(1 + \frac{f_j(p)}{p} + \dots\right) \le \prod_{p > B}\left(1 + \frac{(1+\frac{1}{p})^j - 1}{p} + \left(\frac{p}{p-1}\right)^{j} \frac{1}{p(p-1)}\right)$$

$$\le \exp\left(\int_B^\infty \left(\frac{(1+\frac{1}{x})^j - 1}{x} + \left(\frac{x}{x-1}\right)^{j} \frac{1}{x(x-1)}\right) dx\right).$$

Table 3.1 shows upper bounds of π_j for $j \in [1, 40]$, computed with $A = 500$ and $B = 10^6$. We pick $j = 18$ to get the smallest coefficient in section 2.1.

Table 3.1: Estimates of upper bound of π_j

j	π_j	$\pi_j/2^j$	j	π_j	$\pi_j/2^j$
1	1.23	0.616851	21	52890.17	0.025220
2	1.58	0.394426	22	110675.79	0.026387
3	2.10	0.262036	23	234319.96	0.027933
4	2.90	0.181057	24	501650.18	0.029901
5	4.16	0.130076	25	1085437.39	0.032349
6	6.21	0.097029	26	2372559.18	0.035354
7	9.60	0.074986	27	5236626.72	0.039016
8	15.33	0.059886	28	11666433.49	0.043461
9	25.24	0.049293	29	26225299.18	0.048848
10	42.72	0.041714	30	59464173.31	0.055380
11	74.16	0.036209	31	135959987.61	0.063311
12	131.79	0.032176	32	313376228.98	0.072964
13	239.35	0.029217	33	727960204.98	0.084746
14	443.49	0.027069	34	1703849881.31	0.099177
15	837.30	0.025552	35	4017349516.24	0.116920
16	1608.70	0.024547	36	9539842817.16	0.138823
17	3141.94	0.023971	37	22811315060.41	0.165974
18	**6231.87**	**0.023773**	38	54914363890.79	0.199777
19	12541.54	0.023921	39	133067328413.01	0.242048
20	25588.42	0.024403	40	324514342521.39	0.295144

4 Main argument

4.1 For $n \in \mathcal{A}_k$, $\omega(n) \le \lfloor 4\log k \rfloor$

Let $u = \lceil 4\log k \rceil$; as in Lemma 3.4, we have

$$\sum_{\substack{n\in\mathcal{A}_k\\ \omega(n)\ge u}} \frac{1}{n} \le \sum_{j=u}^{\infty} \frac{1}{j!}\left(\sum_{q\le e^k}\sum_{a=1}^{\infty}\frac{1}{q^a}\right)^j \le \sum_{j=u}^{\infty}\frac{1}{j!}\left(\sum_{q\le e^k}\frac{1}{q-1}\right)^j \le \frac{\mathcal{H}_1(e^k)^u(u+1)}{u!\,(u+1-\mathcal{H}_1(e^k))}.$$

By equations (3.41) and (3.42) in [11], we have that for $n, n' > \exp(K)$,

$$\frac{n'}{n} < \frac{\sigma(n')}{n} = \frac{\sigma(n)}{n} \le \frac{n}{\varphi(n)} < (e^\gamma + \epsilon)\log\log n,$$

and similarly, $n/n' < (e^\gamma+\epsilon)\log\log n'$, where $e^\gamma = 1.78107\ldots$ and $\epsilon \le 0.01$. We have

$$\frac{n}{n'} < (e^\gamma+\epsilon)\log\log n' < (e^\gamma+\epsilon)\left(\log\log(n) + \log\log((e^\gamma+\epsilon)\log\log n)\right)$$

which implies

$$\frac{1}{n'} < \frac{(e^\gamma+\epsilon)\log\log n}{n} + \frac{(e^\gamma+\epsilon)\log\log((e^\gamma+\epsilon)\log\log n))}{n} < \frac{2\log\log n}{n} < \frac{2\log k}{n},$$

for $n > \exp(K)$.

Therefore, we have

$$P^{(\mathbf{1})} \le \sum_{k=K+1}^{\infty} s_k^{(\mathbf{1})},$$

where

$$s_k^{(\mathbf{1})} \le \sum_{k=K+1}^{\infty} (1+2\log k)\cdot\frac{\mathcal{H}_1(e^k)^u(u+1)}{u!\,(u+1-\mathcal{H}_1(e^k))}.$$

We can compute

$$\sum_{k=K+1}^{10^6} s_k^{(\mathbf{1})} \le 0.788795.$$

For $k > 10^6$, $\mathcal{H}_1(e^k) \le \log k + 1.034656$ by Lemma 3.2, and

$$\frac{u+1}{u+1-\mathcal{H}_1(e^k)} = \frac{1}{1-\frac{\mathcal{H}_1(e^k)}{u+1}} \le \frac{1}{1-\frac{\log k + 1.034656}{4\log k + 1}} \le 1.358597.$$

By Stirling's inequality,

$$\frac{1}{u!} \le \frac{1}{\sqrt{2\pi u}}\left(\frac{e}{u}\right)^u.$$

Therefore,

$$E^{(\mathbf{1})} := \sum_{k=10^6+1}^{\infty} s_k^{(\mathbf{1})} \leq 1.358597 \sum_{k=10^6+1}^{\infty} \frac{1+2\log k}{\sqrt{2\pi u}} \left(\frac{e(\log k + 1.034656)}{u}\right)^u$$
$$\leq 1.358597 \sum_{k=10^6+1}^{\infty} \frac{1+2\log k}{\sqrt{8\pi \log k}} \left(\frac{e(\log k + 1.034656)}{4\log k}\right)^{4\log k}$$

because $u \geq 4\log k$ and $\frac{e(\log k+1.034656)}{4\log k} \leq 1$.

For all $k > 10^6$, $\frac{e(\log k+1.034656)}{4\log k} = \frac{e}{4} + \frac{e\times 1.034656}{4\log k} \leq 0.730464 = h$, so we have

$$\begin{aligned}
E^{(\mathbf{1})} &\leq \frac{1.358597}{\sqrt{8\pi\log 10^6}} \sum_{k=10^6+1}^{\infty} (1+2\log k) h^{4\log k} \\
&= 0.072910 \sum_{k=10^6+1}^{\infty} (1+2\log k) k^{4\log h} \\
&\leq 0.072910 \int_{10^6}^{\infty} \frac{1+2\log t}{t^{1.256301}} dt \\
&\leq 0.300429.
\end{aligned}$$

Thus, we have

$$P^{(\mathbf{1})} \leq 0.788795 + 0.300429 = 1.089224.$$

Amicable pair multiplier

For $n \in \mathcal{A}_k, k > K$ satisfying property (1), we will show that

$$\frac{n}{\mu_k} \leq n' \leq n\mu_k,$$

where

$$\mu_k = \prod_{i=2}^{\lceil 4\log k\rceil} \frac{p_i}{p_i - 1}.$$

Proof. If n, n' are even, we have

$$\frac{n'}{n} = \sum_{\substack{d|n\\ d>1}} \frac{1}{d} > \frac{1}{2}.$$

Similarly, $n/n' > 1/2$. The inequality follows immediately since $\mu_k > 2$.

If n and n' are odd, the inequality on the right is straightforward:

$$\frac{n'}{n} = \frac{s(n)}{n} = \frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d} < \prod_{p|n} \frac{p}{p-1} \leq \prod_{i=2}^{\lfloor 4\log k\rfloor+1} \frac{p_i}{p_i-1} = \mu_k.$$

If $n' > n$, then clearly $n' > n/\mu_k$. If $n' \le n$, then $n' \in \mathcal{A}_{k'}$ where $k' \le k$, and it follows that

$$\frac{n}{n'} \le \frac{s(n')}{n'} = \frac{\sigma(n')}{n'} = \prod_{p|n'} \frac{p}{p-1} \le \prod_{i=1}^{\lfloor 4\log k' \rfloor + 1} \frac{p_i}{p_i - 1} < \prod_{i=2}^{\lfloor 4\log k \rfloor + 1} \frac{p_i}{p_i - 1} = \mu_k.$$

□

4.2 For $n \in \mathcal{A}_k$, the largest squarefull divisor of n is at most $L_k/4$

Write $n = sv$ where s is the largest squarefull divisor of n. Assume that $s > L_k/4$; then by Lemma 3.1 and Lemma 3.6,

$$\sum_{\substack{n \in \mathcal{A}_k \\ s > L_k/4}} \left(\frac{1}{n} + \frac{1}{n'}\right) \le (1+\mu_k) \sum_{\substack{n \in \mathcal{A}_k \\ s > L_k/4}} \frac{1}{n} = (1+\mu_k) \sum_{\substack{s > L_k/4 \\ s \text{ squarefull}}} \frac{1}{s} \sum_{\frac{e^{k-1}}{s} < v < \frac{e^k}{s}} \frac{1}{v} \le \frac{3(1+\mu_k)\mathcal{S}_1(L_k/4)}{2}.$$

Therefore,

$$P^{(2)} \le \sum_{k=K+1}^{\infty} \frac{3(1+\mu_k)\mathcal{S}_1(L_k/4)}{2}.$$

By Lemma 3.6, we can compute

$$\sum_{k=K+1}^{10^6} \frac{3(1+\mu_k)\mathcal{S}_1(L_k/4)}{2} \le 35.876404.$$

For $k > 10^6$,

$$\begin{aligned} E^{(2)} := \sum_{k=10^6+1}^{\infty} \frac{3(1+\mu_k)\mathcal{S}_1(L_k/4)}{2} &\le 9 \sum_{k=10^6+1}^{\infty} \frac{1 + 2\log k}{e^{\sqrt{k}/10}} \\ &\le 9 \int_{t=10^6}^{\infty} \frac{1 + 2\log t}{e^{\sqrt{t}/10}} dt \le 1.939049 \times 10^{-37}. \end{aligned}$$

Thus, we have

$$P^{(2)} \le 35.9.$$

4.3 For $n \in \mathcal{A}_k$, the largest squarefree divisor d of n with $P(d) \le L_k$ has $d \le e^{k/3}$

For $n \in \mathcal{A}_k$, write $n = dv$ where d is the largest squarefree divisor such that $P(d) \le L_k$. Assume that $d > e^{k/3}$, we have

$$\sum_{\substack{n \in \mathcal{A}_k \\ d > e^{k/3}}} \frac{1}{n} \le \sum_{\substack{d > e^{k/3} \\ P(d) < L_k \\ d \text{ squarefree}}} \frac{1}{d} \sum_{\frac{e^{k-1}}{d} < v < \frac{e^k}{d}} \frac{1}{v} \le \frac{3}{2} \cdot \frac{\mathcal{H}(L_k)^u}{u!} \cdot \frac{u+1}{u+1-\mathcal{H}(L_k)},$$

where $u = \left\lceil \frac{\log e^{k/3}}{\log L_k} \right\rceil = \left\lceil \frac{k}{3\log L_k} \right\rceil$. Therefore,

$$P^{(\mathbf{3})} \leq \sum_{k=K+1}^{\infty} \frac{3(1+\mu_k)\mathcal{H}(L_k)^u(u+1)}{2u!(u+1-\mathcal{H}(L_k))}.$$

We can compute

$$\sum_{k=K+1}^{10^6} \frac{3(1+\mu_k)\mathcal{H}(L_k)^u(u+1)}{2u!(u+1-\mathcal{H}(L_k))} = 3.622428 \times 10^{-154}.$$

For $k > 10^6$, $\mathcal{H}(L_k) \leq \log\left(\frac{\sqrt{k}}{5}\right) + B + \frac{5}{2k} < \log k$, and $u = \left\lceil \frac{k}{3\log L_k} \right\rceil > \frac{k}{3\log L_k} > \sqrt{k}$, so

$$\frac{u+1}{u+1-\mathcal{H}(L_k)} = \frac{1}{1-\frac{\mathcal{H}(L_k)}{u+1}} \leq \frac{1}{1-\frac{\log k}{\sqrt{k}+1}} < 2.$$

We can estimate

$$\begin{aligned} E^{(\mathbf{3})} &:= \sum_{k=10^6+1}^{\infty} \frac{3(1+\mu_k)\mathcal{H}(L_k)^u(u+1)}{2u!(u+1-\mathcal{H}(L_k))} \leq 3\sum_{k=10^6+1}^{\infty} (1+2\log k)\left(\frac{e\log k}{\sqrt{k}}\right)^{\sqrt{k}} \\ &\leq 3\int_{10^6}^{\infty} (1+2\log t)\left(\frac{e\log t}{\sqrt{t}}\right)^{\sqrt{t}} dt \leq 1.904584 \times 10^{-1421}. \end{aligned}$$

Thus, we have

$$P^{(\mathbf{3})} \leq 3.7 \times 10^{-154}.$$

4.4 For $n \in \mathcal{A}_k$, $P(\gcd(n, s(n))) \leq L_k/2$

Let $r = P(\gcd(n, s(n)))$, and suppose $r > L_k/2$. Because $r|\sigma(n)$, there exists $q^a||n$, such that $r|\sigma(q^a)$. Then $L_k/2 < r \leq \sigma(q^a) < 2q^a$, so $q^a > L_k/4$. It follows from property (2) that $a = 1$, implying that $q \equiv -1 (\text{mod } r)$. Because r is a prime larger than $L_k/2$, $q > r$. We can write $n = rqv$, so that

$$\sum_{\substack{n\in\mathcal{A}_k \\ r>L_k/2}} \frac{1}{n} \leq \sum_{r>L_k/2} \frac{1}{r} \sum_{\substack{r<q\leq e^k \\ q\equiv -1(\text{ mod r})}} \frac{1}{q} \sum_{\frac{e^{k-1}}{qr}<v<\frac{e^k}{qr}} \frac{1}{v} \leq \frac{3}{2} \sum_{r>L_k/2} \frac{1}{r} \sum_{\substack{r<q\leq e^k \\ q\equiv -1(\text{ mod r})}} \frac{1}{q},$$

by Lemma 3.1. From lemma 3.5, we have

$$\sum_{\substack{r<q\leq e^k \\ q\equiv -1(\text{ mod r})}} \frac{1}{q} \leq \frac{2(\log k - \log\log(2-1/r) + 1/\log(e^k/r))}{\varphi(r)} \leq \frac{2(\log k + c)}{r-1},$$

where $c = 2/k - \log\log(2-2/L_k)$ because $L_k/2 < r < \sqrt{qr} \leq \sqrt{n} < e^{k/2}$.

We have

$$\sum_{\substack{n\in\mathcal{A}_k\\ r>L_k/2}} \frac{1}{n} \le 3(\log k + c) \sum_{r>L_k/2} \frac{1}{r(r-1)} \le \frac{3(\log k + c)}{L_k/2 - 1}.$$

Therefore,

$$P^{(4)} \le \sum_{k=K+1}^{\infty} \frac{6(1+\mu_k)(\log k + c)}{L_k - 2}.$$

We can compute

$$\sum_{k=K+1}^{10^6} \frac{6(1+\mu_k)(\log k + c)}{L_k - 2} = 0.050951\ldots.$$

For $k > 10^6$, $2/k - \log\log(2 - 2/L_k) < 1$, so that

$$E^{(4)} \le 6\int_{t=10^6}^{\infty} \frac{(1+2\log t)(\log t + 1)}{e^{\sqrt{t}/5} - 2} dt \le 3.54463 \times 10^{-80}.$$

Thus, we have

$$P^{(4)} \le 0.051.$$

4.5 For $n \in \mathcal{A}_k$, $mm' > \frac{e^k}{L_k}$

In [9], it is shown that each pair m, m' yields at most one amicable pair n, n'. Assume that $mm' \le \frac{e^k}{L_k}$.

If m, m' are both even, then

$$\sum_{\substack{mm' \le \frac{e^k}{L_k}\\ m,m' \text{ even}\\ m<m'}} 1 \le \frac{e^k}{2L_k} \sum_{\substack{m \le \frac{e^k}{L_k}\\ m \text{ even}}} \frac{1}{m} \le \frac{e^k}{4L_k} \sum_{i \le \frac{e^k}{2L_k}} \frac{1}{i} \le \frac{e^k}{L_k} \cdot \frac{k - \log L_k + 1 - \log 2}{4}.$$

If m, m' are both odd, because pm, $p'm'$ is an amicable pair, exactly one of pm, $p'm'$ is abundant, call it $\hat{p}\hat{m}$. Then, $(\hat{p}+1)\sigma(\hat{m}) > 2\hat{p}\hat{m}$, so $h(\hat{m}) > 2\hat{p}/(\hat{p}+1) > 2 - \epsilon$, for some small $\epsilon = \frac{2}{\hat{p}+1}$ because $\hat{p}$ is big. In particular, $\hat{p} > e^{(\hat{k}-1)/\lfloor 4\log\hat{k}\rfloor}$, by properties (1), (2), (3). We have

$$\sum_{\substack{mm' \le \frac{e^k}{L_k}\\ m,m' \text{ odd}}} 1 = \frac{e^k}{L_k} \sum_{\substack{\hat{m} \le \frac{e^k}{L_k}\\ \hat{m} \text{ odd}\\ h(\hat{m}) \ge 2-\epsilon}} \frac{1}{\hat{m}} \le \frac{e^k}{L_k} \cdot \frac{\pi_j(k - \log L_k + 2.307)}{2(2-\epsilon)^j}.$$

Therefore, we have

$$\sum_{\substack{n\in\mathcal{A}_k\\ mm' \le \frac{e^k}{L_k}}} \frac{1}{n} \le \frac{1}{e^{k-1}} \sum_{mm' \le e^k/L_k} 1 \le \frac{1}{e^{k-1}} \left(\sum_{\substack{mm' \le \frac{e^k}{L_k}\\ m,m' \text{ odd}}} 1 + \sum_{\substack{mm' \le \frac{e^k}{L_k}\\ m,m' \text{ even}}} 1 \right).$$

Since we are picking up all pairs of m, m' such that $m, m' \leq e^k/L_k$ without any constraint as to in which intervals n and n' fall, this part does not need a multiplier. In fact, we not only pick up both n and n' in the argument, but also may have counted each pair multiple times, thus

$$P^{(5)} \leq \sum_{k=K+1}^{\infty} \frac{e}{L_k}\left(\frac{\pi_j/2(k-\log L_k+2.307)}{(2-\epsilon)^j}+\frac{k-\log L_k+1-\log 2}{4}\right).$$

Choosing $j = 18$, we can compute

$$\sum_{k=K+1}^{10^6} \frac{e}{L_k}\left(\frac{\pi_j(k-\log L_k+2.307)}{2(2-\epsilon)^j}+\frac{k-\log L_k+1-\log 2}{4}\right) = 0.806677\ldots.$$

For $k > 10^6$, we can ignore the parity of m, m' and estimate

$$E^{(5)} \leq e\sum_{k=10^6+1}^{\infty} \frac{k-\log(L_k)+1-\log 2}{L_k} < e\int_{t=10^6+1}^{\infty} \frac{t+1}{e^{\sqrt{t}/5}}dt \leq 3.818819\times 10^{-77}.$$

Thus,

$$P^{(5)} \leq 0.807.$$

4.6 For $n \in \mathcal{A}_k$, $p \leq e^{3k/4}L_k$

Suppose that $n \in \mathcal{A}_k$ and $p > e^{3k/4}L_k$, then $m < e^{k/4}/L_k$. By property (5), $m' > e^{3k/4}$. Since $n' < \mu_k e^k$, it follows that $p' < \mu_k e^{k/4}$.

We can write $n' = p'_1p'_2\ldots p'_js'$ where s' is the largest squarefull divisor and $\{p'_i\}$ are in descending order. Since n' satisfies property (2), we have

$$p'_1p'_2\ldots p'_j = \frac{n'}{s'} > \frac{e^{k-1}}{\mu_k}\cdot\frac{4}{L_{k'}} > \frac{e^k}{\mu_k L_k} > e^{k/2}.$$

Therefore, we can pick i to be the largest such that $p'_1\ldots p'_{i-1} < e^{k/2}$.

Let $D = p'_1\ldots p'_i$. Write $n' = DM$, so that $\gcd(D, M) = 1$ and $M < e^k\mu_k/D$. Furthermore, we have

$$e^{k/2} < D = (p'_1p'_2\ldots p'_{i-1})p'_i < e^{3k/4}\mu_k.$$

Recall from [9] that such m satisfy the following congruence

$$\sigma(m)DM \equiv m\sigma(m) \pmod{\sigma(D)}.$$

The number of choices for $M < e^k\mu_k/D$ which satisfy this congruence is at most

$$1+\frac{e^k\mu_k}{D\sigma(D)/\gcd(\sigma(m)D,\sigma(D))} \leq 1+\frac{e^k\mu_k\sigma(m)\gcd(D,\sigma(D))}{D^2}.$$

Every prime dividing D exceeds $L_k/2$, because

$$p'_i\ldots p'_j = \frac{n'}{p'_1\ldots p'_{i-1}s'} > \frac{e^{k/2-1}}{\mu_k}\cdot\frac{4}{L_{k'}} > e^{k'/3},$$

and n' has property (3), so it follows that $p'_i > L_{k'} > L_k/2$.

Since $\gcd(D, \sigma(D)) | (n, n')$, property (4) implies that $\gcd(D, \sigma(D)) = 1$. Also, $\sigma(m) \leq m\mu_k < e^{k/4}\mu_k/L_k$. Given the choice of m, D, the number of choices for M is at most:

$$1 + \frac{e^{5k/4}\mu_k^2}{L_k D^2}.$$

Therefore, we have

$$\begin{aligned}
\sum_{\substack{n \in \mathcal{A}_k \\ p > e^{3k/4} L_k}} 1 &\leq \sum_{e^{k/2} < D \leq e^{3k/4}\mu_k} \left(1 + \frac{e^{5k/4}\mu_k^2}{L_k D^2}\right) \\
&\leq e^{3k/4}\mu_k + \frac{e^{5k/4}\mu_k^2}{L_k} \sum_{D > e^{k/2}\mu_k} \frac{1}{D^2} \\
&\leq e^{3k/4}\mu_k + \frac{e^{5k/4}\mu_k^2}{L_k} \left(\frac{1}{e^k} + \int_{e^{k/2}}^{\infty} \frac{1}{t^2} dt\right) \\
&\leq e^{3k/4}\mu_k + \frac{e^{5k/4}\mu_k^2}{L_k} \left(\frac{1}{e^k} + \frac{1}{e^{k/2}}\right) \\
&\leq e^{3k/4}\mu_k + \frac{e^{k/4}\mu_k^2}{L_k} + \frac{e^{3k/4}\mu_k^2}{L_k}.
\end{aligned}$$

Thus, the reciprocal sums of such numbers are at most

$$\sum_{\substack{n \in \mathcal{A}_k \\ p > e^{3k/4} L_k}} \frac{1}{n} \leq \frac{a_k^{(6)}}{e^{k-1}} \leq e\left(\frac{\mu_k}{e^{k/4}} + \frac{\mu_k^2}{L_k e^{3k/4}} + \frac{\mu_k^2}{L_k e^{k/4}}\right),$$

so that

$$P^{(\mathbf{6})} \leq e \sum_{k=K+1}^{\infty} (1 + \mu_k) \left(\frac{\mu_k}{e^{k/4}} + \frac{\mu_k^2}{L_k e^{3k/4}} + \frac{\mu_k^2}{L_k e^{k/4}}\right).$$

Note that the denominator contains $e^{k/4}$, which is significantly larger than the numerator. In particular, we have

$$P^{(\mathbf{6})} \leq 3e \sum_{k=K+1}^{\infty} \frac{(1 + 2\log k)^3}{e^{k/4}} \leq 3e \int_K^{\infty} \frac{(1 + 2\log t)^3}{e^{t/4}} dt \leq 2.5 \times 10^{-538}.$$

4.7 For $n \in \mathcal{A}_k$, $q_0 \leq 16L_k$

Recall that for $n = pm \in \mathcal{A}_k$, we can write $m = m_0 m_1$ where m_0 is the largest divisor of m such that m_0 is either a squarefull number or twice one. Similarly, we can write $q+1 = q_0 q_1$, where q_0 is the largest divisor of $q + 1$ such that q_0 is either a squarefull number or twice one. Assume that $q_0 > 16L_k$. Because $q < p$, it follows that $q < e^{k/2}$, so we have

$$\sum_{\substack{n \in \mathcal{A} \\ q_0 > 16L_k}} \frac{1}{n} \leq \sum_{q_0 > 16L_k} \sum_{\substack{q \equiv -1 \pmod{q_0} \\ q < e^{k/2}}} \frac{1}{q} \sum_{e^{k-1}/q \leq v \leq e^k/q} \frac{1}{v} \leq \iota \sum_{q_0 > 16L_k} \sum_{\substack{q \equiv -1 \pmod{q_0} \\ q < e^{k/2}}} \frac{1}{q}$$

where $\iota = 1 + 1/e^{k/2-1}$.

If $q_0 > e^{k/4}$, then $q > e^{k/4} - 1$, so that

$$\sum_{\substack{q \equiv -1 \pmod{q_0} \\ q < e^{k/2}}} \frac{1}{q} \le \iota' \sum_{\substack{q_0 \mid (q+1) \\ q < e^{k/2}}} \frac{1}{q+1} = \frac{\iota'}{q_0} \sum_{i=1}^{\left\lfloor \frac{e^{k/2}}{q_0} \right\rfloor} \frac{1}{i} \le \frac{\iota'(1+k/4)}{q_0},$$

where $\iota' = 1 + 1/(e^{k/4} - 1)$. Therefore, we have

$$\sum_{q_0 > e^{k/4}} \sum_{\substack{q \equiv -1 \pmod{q_0} \\ q < e^{k/2}}} \frac{1}{q} \le \iota'(1+k/4) \sum_{q_0 > e^{k/4}} \frac{1}{q_0}.$$

Because $q_0 q_1$ is even, the definition of q_0 implies that q_0 is either an even squarefull number or twice an odd squarefull number, i.e., $2q_0$ is squarefull. We have

$$\sum_{\substack{q_0 > e^{k/4} \\ 2q_0 \text{ squarefull}}} \frac{1}{q_0} = 2 \sum_{\substack{i > 2e^{k/4} \\ i \text{ squarefull}}} \frac{1}{i} = 2\mathcal{S}_1(2e^{k/4}).$$

If $16L_k < q_0 \le e^{k/4}$, then by Lemma 3.5, we have

$$\begin{aligned}
\sum_{\substack{q \equiv -1 \pmod{q_0} \\ q < e^{k/2}}} \frac{1}{q} &\le \frac{1}{q_0 - 1} + \frac{2\left(\log\log(e^{k/2}) - \log\log(2 - 1/q_0) + \frac{1}{\log(e^{k/2}/q_0)}\right)}{\varphi(q_0)} \\
&\le \frac{1}{q_0 - 1} + \frac{2\log k + c}{\varphi(q_0)},
\end{aligned}$$

where $c = 8/k - 2\log\log(2 - 1/16L_k)$. Therefore,

$$\sum_{16L_k < q_0 < e^{k/4}} \sum_{\substack{q \equiv -1 \pmod{q_0} \\ q < e^{k/2}}} \frac{1}{q} \le \left(\sum_{\substack{q_0 > 16L_k \\ 2q_0 \text{ squarefull}}} \frac{1}{q_0 - 1} \right) + (2\log k + c) \left(\sum_{\substack{q_0 > 16L_k \\ 2q_0 \text{ squarefull}}} \frac{1}{\varphi(q_0)} \right).$$

By Lemma 3.6, we have

$$\sum_{\substack{q_0 > 16L_k \\ 2q_0 \text{ squarefull}}} \frac{1}{q_0 - 1} \le \iota'' \sum_{\substack{q_0 > 16L_k \\ 2q_0 \text{ squarefull}}} \frac{1}{q_0} \le 2\iota''\mathcal{S}_1(32L_k),$$

where $\iota'' = 1 + 1/(16L_k - 1)$, and, using equation (3.41) in [11],

$$\sum_{16L_k < q_0 < e^{k/2}} \frac{1}{\varphi(q_0)} \le \sum_{\substack{q_0 > 16L_k \\ 2q_0 \text{ squarefull}}} \frac{1}{q_0} \left(e^{\gamma} \log\log q_0 + \frac{5}{2\log\log q_0} \right)$$

$$\leq e^\gamma \sum_{\substack{q_0>16L_k\\ 2q_0 \text{ squarefull}}} \frac{\log\log q_0}{q_0} + \frac{5}{2\log\log 16L_k} \sum_{\substack{q_0>16L_k\\ 2q_0 \text{ squarefull}}} \frac{1}{q_0}$$
$$\leq 2e^\gamma \sum_{\substack{i>32L_k\\ i \text{ squarefull}}} \frac{\log\log i}{i} + \frac{5}{\log\log 16L_k} \sum_{\substack{i>32L_k\\ i \text{ squarefull}}} \frac{1}{i}$$
$$\leq 2e^\gamma \mathcal{S}_*(32L_k) + \frac{5\mathcal{S}_1(32L_k)}{\log\log 16L_k},$$

where $e^\gamma = 1.78107\ldots$.

Thus, we have

$$\sum_{\substack{n\in\mathcal{A}_k\\ q_0>16L_k}} \left(\frac{1}{n}+\frac{1}{n'}\right) \leq (1+\mu_k)\iota\Bigg[2\iota'(1+k/4)\mathcal{S}_1(2e^{k/4}) + 2\iota''\mathcal{S}_1(32L_k)$$
$$+(2\log k + c)\left(2e^\gamma\mathcal{S}_*(32L_k) + \frac{5\mathcal{S}_1(32L_k)}{\log\log 16L_k}\right)\Bigg].$$

Using Lemma 3.6, we can compute

$$\sum_{k=K+1}^{10^6} s_k^{(\mathbf{7})} = 642.0724\ldots.$$

For $k > 10^6$, we have $c < 1$, $\iota, \iota', \iota'' < 2$, so that

$$E^{(\mathbf{7})} \leq 2\sum_{k=10^6+1}^{\infty} (1+2\log k)\left(\frac{12(1+k/4)}{e^{k/8}} + \frac{3}{e^{\sqrt{k}/10}} + (2\log k+1)\left(\frac{4\log k}{e^{\sqrt{k}/10}} + \frac{15}{e^{\sqrt{k}/5}}\right)\right)$$
$$\leq 2\int_{10^6}^{\infty} (1+2\log t)\left(\frac{12(1+t/4)}{e^{t/8}} + \frac{3}{e^{\sqrt{t}/10}} + (2\log t+1)\left(\frac{4\log t}{e^{\sqrt{t}/10}} + \frac{15}{e^{\sqrt{t}/5}}\right)\right) dt$$
$$\leq 6.85011 \times 10^{-35}.$$

Thus, we have

$$P^{(\mathbf{7})} \leq 642.073.$$

4.8 For $n \in \mathcal{A}_k$, $P(\sigma(m_1)) \geq L_k$

Assume $P(\sigma(m_1)) < L_k$, then $P(q+1) = P(\sigma(q)) \leq P(\sigma(m_1)) < L_k$.
We have $\omega(m_1) \leq \lfloor 4\log k \rfloor - 1$ by property (1), and

$$m_1 = \frac{n}{pm_0} \geq \frac{2e^{k/4-1}}{L_k^2},$$

by property (6) and since $m_0 \leq 2s \leq L_k/2$ by property (2), so that

$$q \geq \left(\frac{2e^{k/4-1}}{L_k^2}\right)^{1/(\lfloor 4\log k\rfloor -1)} = Q_0, \text{ say.}$$

Let $Q_i = Q_0 L_k^i$, and consider $q \in [Q_i, Q_{i+1})$ for separate values of i. We have $q+1 = q_0 q_1$ as in the last section, and

$$\sum_{\substack{n\in\mathcal{A}_k\\ P(\sigma(m_1))<L_k\\ q\in[Q_i,Q_{i+1})}} \frac{1}{n} \le \iota_i \sum_{q_0\le 16L_k} \frac{1}{q_0} \sum_{m_0\le L_k/2} \frac{1}{m_0} \sum_{\substack{q_1\ge \frac{Q_i}{16L_k}\\ P(q_1)\le L_k}} \frac{1}{q_1} \sum_{\substack{m_2\ge \frac{2e^{k/4-1}}{Q_{i+1}L_k^2}\\ P(m_2)<Q_{i+1}}} \frac{1}{m_2} \sum_{\substack{p>Q_i\\ z\le p\le ez}} \frac{1}{p},$$

where $\iota_i = 1 + 1/Q_i$, and $z = e^{k-1}/qm_0m_2$. Since q_0 is always even,

$$\sum_{q_0} \frac{1}{q_0} = \left(\frac{1}{2} + \frac{1}{4} + \dots\right) \prod_{p>2} \left(1 + \frac{1}{p(p-1)}\right) = \frac{2\zeta(2)\zeta(3)}{3\zeta(6)}.$$

If n, n' are both even, then m_0 is even, and

$$\sum_{m_0} \frac{1}{m_0} = \frac{2\zeta(2)\zeta(3)}{3\zeta(6)}.$$

If n, n' are odd, then m_0 is odd, and

$$\sum_{m_0} \frac{1}{m_0} = \prod_{p>2} \left(1 + \frac{1}{p(p-1)}\right) = \frac{2\zeta(2)\zeta(3)}{3\zeta(6)}.$$

Thus,

$$\begin{aligned} g_i := \sum_{\substack{n\in\mathcal{A}_k\\ P(\sigma(m_1))<L_k\\ q\in[Q_i,Q_{i+1})}} \left(\frac{1}{n} + \frac{1}{n'}\right) &\le \sum_{\substack{n\in\mathcal{A}_k\\ P(\sigma(m_1))<L_k\\ q\in[Q_i,Q_{i+1})\\ n,n' \text{ even}}} \left(\frac{1}{n} + \frac{1}{n'}\right) + \sum_{\substack{n\in\mathcal{A}_k\\ P(\sigma(m_1))<L_k\\ q\in[Q_i,Q_{i+1})\\ n,n' \text{ odd}}} \left(\frac{1}{n} + \frac{1}{n'}\right) \\ &\le (4+\mu_k)\iota_i \left(\frac{2\zeta(2)\zeta(3)}{3\zeta(6)}\right)^2 \sum_{\substack{q_1\ge \frac{Q_i}{16L_k}\\ P(q_1)\le L_k}} \frac{1}{q_1} \sum_{\substack{m_2\ge \frac{2e^{k/4-1}}{Q_{i+1}L_k^2}\\ P(m_2)<Q_{i+1}}} \frac{1}{m_2} \sum_{\substack{p>Q_i\\ z\le p\le ez}} \frac{1}{p}, \end{aligned}$$

since the multiplier for even pair is 3 and for odd pair is $1 + \mu_k$.

Here, $\omega(q_1) \ge \left\lceil \frac{\log(Q_i/16L_k)}{\log L_k} \right\rceil = u_i$, say, so as in Lemma 3.4, we have

$$\sum_{\substack{q_1> \frac{Q_i}{16L_k}\\ P(q_1)\le L_k}} \frac{1}{q_1} \le \sum_{\substack{P(q_1)<L_k\\ \omega(q_1)\ge u_i}} \frac{1}{q_1} \le \frac{(\mathcal{H}(L_k) - 1/2)^{u_i}}{u_i!} \cdot \frac{u_i+1}{u_i + 1/2 - \mathcal{H}(L_k)}, \tag{4.1}$$

since q_1 runs over odd squarefree numbers.

Similarly, $\omega(m_2) \ge \left\lceil \frac{k/4-1+\log 2-\log Q_{i+1}-2\log L_k}{\log Q_{i+1}} \right\rceil = w_i$, say, so

$$\sum_{\substack{m_2\ge \frac{2e^{k/4-1}}{Q_{i+1}L_k^2}\\ P(m_2)<Q_{i+1}}} \frac{1}{m_2} \le \sum_{\substack{P(m_2)<Q_{i+1}\\ \omega(m_2)\ge w_i}} \frac{1}{m_2} \le \frac{(\mathcal{H}(Q_{i+1}) - 1/2)^{w_i}}{w_i!} \cdot \frac{w_i+1}{w_i + 1/2 - \mathcal{H}(Q_{i+1})}. \tag{4.2}$$

We also have

$$\sum_{\substack{p>Q_i\\ z\le p\le ez}} \frac{1}{p} \le \frac{1}{\log Q_i} + \frac{1}{\log^2 Q_i}.$$

Note that estimates (4.1) and (4.2) are valid only if $u_i + 1/2 > \mathcal{H}(L_k)$ and $w_i + 1/2 > \mathcal{H}(Q_{i+1})$. For small values of i, u_i will be too small to use (4.1) and for large values of i, w_i will be too small to use (4.2). Let $[a, b]$ be the interval of indices i where the estimates (4.1) and (4.2) are valid. For $q < Q_a$, we have

$$\begin{aligned}\alpha = \sum_{\substack{n\in\mathcal{A}_k\\ P(\sigma(m_1))<L_k\\ q<Q_a}} \left(\frac{1}{n} + \frac{1}{n'}\right) &\le (1+\mu_k)\iota_0 \sum_{\substack{m_2\ge \frac{2e^{k/4-1}}{Q_a L_k^2}\\ P(m_2)<Q_a}} \frac{1}{m_2} \sum_{\substack{z\le v\le ez\\ z>Q_0}} \frac{1}{v}\\ &\le (1+\mu_k)\iota_0^2 \cdot \frac{(\mathcal{H}(Q_a)-1/2)^{w_{a-1}}}{w_{a-1}!} \cdot \frac{w_{a-1}+1}{w_{a-1}+1/2-\mathcal{H}(Q_a)},\end{aligned}$$

where $\iota_0 = 1 + 1/Q_0$. For $q > Q_b$, we have

$$\begin{aligned}\beta = \sum_{\substack{n\in\mathcal{A}_k\\ P(\sigma(m_1))<L_k\\ q\ge Q_b}} \left(\frac{1}{n} + \frac{1}{n'}\right) &\le (1+\mu_k)\iota_b \sum_{\substack{q_1>\frac{Q_b}{16L_k}\\ P(q_1)\le L_k}} \frac{1}{q_1} \sum_{\substack{z\le v\le ez\\ z>Q_b}} \frac{1}{v}\\ &\le (1+\mu_k)\iota_b^2 \cdot \frac{(\mathcal{H}(L_k)-1/2)^{u_b}}{u_b!} \cdot \frac{u_b+1}{u_b+1/2-\mathcal{H}(L_k)},\end{aligned}$$

where $\iota_b = 1 + 1/Q_b$. Thus, we have

$$s_k^{(\mathbf{8})} = \sum_{\substack{n\in\mathcal{A}_k\\ P(\sigma(m_1))<L_k}} \left(\frac{1}{n} + \frac{1}{n'}\right) \le \alpha + \beta + \sum_{i=a}^{b-1} g_i.$$

We can compute

$$\sum_{k=K+1}^{10^6} s_k^{(\mathbf{8})} = 306.2117.$$

For $k > 10^6$, we use $a = b = 0$, so that

$$\mathcal{H}(L_k) - 1/2 < \frac{\log k}{2} - \log 5 + B + \frac{5}{2k} - 1/2 < \frac{\log k}{2} - 1.847941,$$

and

$$\begin{aligned}u_0 = \left\lceil \frac{\log(Q_0/16L_k)}{\log L_k} \right\rceil &> \frac{\log Q_0 - \log 16 - \sqrt{k}/5}{\sqrt{k}/5}\\ &\ge \frac{\frac{k/4-1+\log 2-2\sqrt{k}/5}{4\log k-1} - \log 16 - \sqrt{k}/5}{\sqrt{k}/5} > 0.219k^{1/3},\end{aligned}$$

which implies

$$\begin{aligned}
E^{(\mathbf{8})} &\leq \sum_{k=10^6+1}^{\infty} \frac{1+2\log k}{\sqrt{2\pi \cdot 0.219 \cdot k^{1/3}}} \left(\frac{e(\log k/2 - 1.847941)}{0.219 \cdot k^{1/3}} \right)^{0.219 \cdot k^{1/3}} \\
&\leq \int_{10^6}^{\infty} \frac{1+2\log t}{\sqrt{2\pi \cdot 0.219 \cdot t^{1/3}}} \left(\frac{e(\log t/2 - 1.847941)}{0.219 \cdot t^{1/3}} \right)^{0.219 \cdot t^{1/3}} dt \\
&\leq 11.2041.
\end{aligned}$$

Thus, we have

$$P^{(\mathbf{8})} \leq 317.4158.$$

4.9 Remaining amicable numbers

We are left with amicables n such that both n and n' have properties (1)–(8). We want to calculate

$$P^{(*)} = \sum_{\substack{n\in\mathcal{A}\\ \min\{n,n'\}>K\\ n,n' \text{ pass (1)-(8)}}} \frac{1}{n} \leq \sum_{k=K+1}^{\infty}(1+\mu_k) \sum_{\substack{n\in\mathcal{A}_k\\ n,n' \text{ pass (1)-(8)}\\ P(n)>P(n')}} \frac{1}{n}.$$

By property 8, there exists a prime $r|\sigma(m_1)$ with $r > L_k$. Thus, there is a prime $q|m$ with $q \equiv -1(\text{mod } r)$. But $\sigma(n) = \sigma(s(n))$, so there is a prime power $q'^a||s(n)$ with $r|\sigma(q'^a)$. Then $q'^a > r/2 > L_k/2 > L_{k'}/4$, so by property 2, $a = 1$ and $q' \equiv -1(\text{mod } r)$. Since $q' > L_k/2$, by part 4 we have $q' \nmid n$, so $q' \neq q$. Also, $q'|s(n)$ implies that

$$s(n) = ps(m) + \sigma(m) \equiv 0(\text{mod } q').$$

Since $q' \nmid n$, it implies that $q' \nmid \sigma(m)$, so p is in a residue class $a = a(m,q')(\text{mod } q')$. And since $P(n) > P(n')$, $p > q'$.

We want to calculate the following:

$$\sum_{\substack{n\in\mathcal{A}_k\\ n,n' \text{ pass (1)-(8)}\\ P(n)>P(n')}} \frac{1}{n} \leq \sum_{r>L_k} \sum_{\substack{q\equiv-1(\text{mod } r)\\ q<e^k}} \sum_{\substack{m\equiv0(\text{mod } q)\\ m<e^k}} \frac{1}{m} \sum_{\substack{q'\equiv-1(\text{mod } r)\\ q'<e^k}} \sum_{\substack{p\equiv a(\text{mod } q')\\ q'<p\leq e^k/m}} \frac{1}{p}.$$

Since $q' \equiv -1(\text{mod } r)$, we have $q' > r \geq L_k$. Let $\iota = \frac{L_k}{L_k-1}$, then $1/\varphi(r) \leq \iota/r$ and $1/\varphi(q') \leq \iota/q'$.

If $q' > \frac{e^{k-1}}{m}$, then $p \in \{q'+a, 2q'+a\}$, and since $(q'+a)$ and $(2q'+a)$ are of different parity, there is at most one choice of p. Writing $m = qj$, then $e^{k-1}/pq \leq j \leq e^k/pq$, so $\sum_j 1/j \leq 3/2$. Therefore,

$$\sum_{\substack{n\in\mathcal{A}_k\\ n,n' \text{ pass (1)-(8)}\\ P(n)>P(n')\\ q'>\frac{e^{k-1}}{m}}} \frac{1}{n} \leq \frac{3}{2} \sum_{r>L_k} \sum_{\substack{q\equiv-1(\text{mod } r)\\ q<e^k}} \frac{1}{q} \sum_{\substack{q'\equiv-1(\text{mod } r)\\ q'<e^k}} \frac{1}{q'}.$$

If $q' \leq \frac{e^{k-1}}{m}$, then $j \leq \frac{e^{k-1}}{qq'}$, so that

$$\sum_{\substack{n\in\mathcal{A}_k\\ n,n' \text{ pass (1)-(8)}\\ P(n)>P(n')\\ q'\leq\frac{e^{k-1}}{m}}} 1 \leq \sum_{r>L_k} \sum_{\substack{q\equiv-1(\text{mod } r)\\ q<e^k}} \sum_{\substack{q'\equiv-1(\text{mod } r)\\ q'<e^k}} \sum_{j\leq\frac{e^{k-1}}{qq'}} \sum_{\substack{p\equiv a(\text{mod } q')\\ p\leq e^k/jq}} 1$$

$$\leq \sum_{r>L_k} \sum_{\substack{q\equiv-1(\text{mod } r)\\ q<e^k}} \sum_{\substack{q'\equiv-1(\text{mod } r)\\ q'<e^k}} \sum_{j\leq\frac{e^{k-1}}{qq'}} \frac{2\iota e^k}{q'qj\log(e^k/q'qj)}$$

$$\leq 2\iota e^k \sum_{r>L_k} \sum_{\substack{q\equiv -1 \pmod r \\ q<e^k}} \frac{1}{q} \sum_{\substack{q'\equiv -1 \pmod r \\ q'<e^k}} \frac{1}{q'} \sum_{j\leq z/e} \frac{1}{j\log(z/j)},$$

where $z = \frac{e^k}{qq'}$. Since $qq' \leq mq' \leq e^{k-1}$, $z \geq e$. We have

$$\sum_{j\leq z/e} \frac{1}{j\log(z/j)} \leq \frac{1}{\log z} + \int_1^{z/e} \frac{dt}{t\log(z/t)} = \frac{1}{\log z} - \log\log\frac{z}{t}\Big|_1^{z/e}$$
$$= \frac{1}{\log z} + \log\log z < 1 + \log k,$$

so

$$\sum_{\substack{n\in\mathcal{A}_k \\ n,n' \text{ pass (1)-(8)} \\ P(n)>P(n') \\ q'\leq \frac{e^{k-1}}{m}}} \frac{1}{n} = 2\iota e(1+\log k) \sum_{r>L_k} \sum_{\substack{q\equiv -1 \pmod r \\ q<e^k}} \frac{1}{q} \sum_{\substack{q'\equiv -1 \pmod r \\ q'<e^k}} \frac{1}{q'}.$$

Thus,

$$\sum_{\substack{n\in\mathcal{A}_k \\ n,n' \text{ pass (1)-(8)} \\ P(n)>P(n')}} \frac{1}{n} \leq \left(2\iota e(1+\log k) + \frac{3}{2}\right) \sum_{r>L_k} \sum_{\substack{q\equiv -1 \pmod r \\ q<e^k}} \frac{1}{q} \sum_{\substack{q'\equiv -1 \pmod r \\ q'<e^k}} \frac{1}{q'}.$$

By Lemma 3.5, we have

$$\sum_{\substack{q\equiv -1 \pmod r \\ q\leq e^k}} \frac{1}{q} \leq \frac{2(\log k + c)}{\varphi(r)} \leq \frac{2\iota(\log k + c)}{r},$$

where $c = \frac{1}{k/4 - \log L_k} - \log\log(2 - 1/L_k)$ because $L_k \leq r < p \leq e^{3k/4}L_k$. Therefore,

$$\sum_{r>L_k} \sum_{\substack{q\equiv -1 \pmod r \\ q<e^k}} \frac{1}{q} \sum_{\substack{q'\equiv -1 \pmod r \\ q'<e^k}} \frac{1}{q'} \leq 4\iota^2(\log k + c)^2 \sum_{r\geq L_k} \frac{1}{r^2} \leq \frac{4\iota^2(\log k + c)^2}{L_k - 1}.$$

Thus,

$$s_k^{(*)} \leq \frac{4\iota^3(1+\mu_k)\left(2\iota e(1+\log k) + 3/2\right)(\log k + c)^2}{L_k}.$$

We can compute

$$\sum_{k=K+1}^{10^6} s_k^{(*)} \leq 17.1315.$$

For $k > 10^6$, $c < 1$, and choose a large factor, say 30, to offset the constants, we have

$$E^{(*)} \leq 30 \sum_{k=10^6+1}^{\infty} \frac{(1+2\log k)(1+\log k)^3}{e^{\sqrt{k}/5}}$$

$$\le 30 \int_{10^6}^{\infty} \frac{(1 + 2\log t)(1 + \log t)^3}{e^{\sqrt{t}/5}}$$
$$\le 3.89548 \times 10^{-77}.$$

Thus,

$$P^{(*)} \le 17.1316.$$

Putting everything together, we have the result stated in Theorem 1.1:

$$P = P_s + P(K) + P^{(*)} + \sum_{\mathbf{i}=1}^{8} P^{(\mathbf{i})} + \varepsilon < 4084.$$

This upper bound can potentially be decreased further by (1) extending the parity argument currently used for small amicable numbers to address large amicable numbers, and (2) break down part 4.8 even further by computing small cases of q_0 and m_0.

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