Pattern Avoidance in Nonnesting Permutations

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Abstract

Nonnesting permutations are the permutations of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ that avoid the pattern $ijji$. They are studied in connection with noncrossing permutations, also known as quasi-Stirling permutations, which avoid the pattern $ijij$ and generalize the well-known Stirling permutations. Inspired by the work of Archer et al. on pattern avoidance in noncrossing permutations, we extend the result to the nonnesting case. Specifically, we enumerate nonnesting permutations that avoid each set of at least two patterns of length 3, as well as other patterns, obtaining a closed formula in each enumeration. Most proofs use recurrences.

1 Introduction

Denote by $[n] \sqcup [n] = \{1, 1, 2, 2, ..., n, n\}$ the multiset consisting of two copies of each integer from 1 to n. We denote the set of all permutations on the multiset $[n] \sqcup [n]$ by $\mathcal{S}_n^2$.

Let $\pi = \pi_1\pi_2...\pi_n$ and $\sigma = \sigma_1\sigma_2...\sigma_k$ be permutations over the positive integer $\mathbb{N}$. We say π *contains* σ if there is some $1 \leq i_1 < i_2 < ... < i_k \leq n$ such that the subsequence $\pi_{i_1}\pi_{i_2}...\pi_{i_k}$ and σ have the same relative order, that is,

- $\pi_{i_r} = \pi_{i_s}$ if and only if $\sigma_r = \sigma_s$, and
- $\pi_{i_r} < \pi_{i_s}$ if and only if $\sigma_r < \sigma_s$

for all $1 \leq i, j \leq k$. In this case, we call this subsequence an *occurrence* of σ.

If π does not contain σ, we say that π *avoids* σ.

A *nonnesting permutation* $\pi \in \mathcal{S}_n^2$ is a permutation that avoids the nesting patterns 1221 and 2112, that is, there does not exist indices $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ such that $i_1 = i_4$ and $i_2 = i_3$. Similarly, we define a *noncrossing permutation* as one that avoids the crossing patterns 1212 and 2121. Archer et al. [1] introduced noncrossing permutations, also called *quasi-Stirling permutations* in their paper, as a generalization of Stirling permutations [4].

One can understand nonnesting permutations graphically as follows. Given a permutation $\pi \in \mathcal{S}_n^2$, connect π_i and π_j with an arc labeled l if $\pi_i = \pi_j = l$. Then a nonnesting permutation is a nonnesting matching with distinct labels from $[n]$. Similarly, a noncrossing permutation is a labeled matching whose arcs do not cross each other. In fact, the nonnesting condition ensures that the order of the left endpoints of the arc is the same that of the right endpoints. The permutation obtained from reading the labels of the arcs from left to right is called the *underlying permutation*.

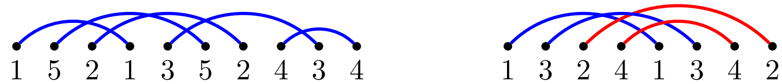

Figure 1: The permutation 1523415234 is nonnesting, but the permutation 13241342 is not.

Denote $\mathcal{C}_n$ the set of nonnesting permutations on the multiset $\{1, 1, 2, 2, ..., n, n\}$. Write $|\mathcal{C}_n| = s_n$. It is well-known that the number of nonnesting or noncrossing matchings with n arcs is given by the nth Catalan number Cat_n. Since there are $n!$ ways to label the arcs, the number of nonnesting and the number of noncrossing permtuations are counted by

$$n!\,\mathrm{Cat}_n = \frac{(2n)!}{(n+1)!}[3].$$

Archer et al. [1] enumerated noncrossing permutations that avoid at least two patterns of length 3. Athanasiadis constructed a bijection between noncrossing partition and nonnesting partition [2], which can be extended to a bijection between noncrossing permutation and nonnesting permutation. Motivated by these two papers, we extend Archer et al.'s enumeration to the nonnesting case. In particular, this research considers the nonnesting permutations that avoid subsets of permutations of $\{1, 2, 3\}$.

Consider $\pi = \pi_1\pi_2...\pi_n \in \mathcal{C}_n(\Lambda)$. We say the reverse of π is $\pi^r = \pi_n\pi_{n-1}...\pi_1$, and the complement of π is π^c where $\pi^c_i = n + 1 - \pi_i$. Let τ be a pattern over the positive integers. Then π avoids τ if and only if π^r avoids τ^r if and only if π^c avoids τ^c.

2 Result

In this section, we enumerate nonnesting permutations that avoid at least two patterns of length 3. Under complementation and reversal, Table 1 provides all possible enumeration of $\mathcal{C}_n(\Lambda)$ that are not trivially equivalent, $\Lambda \subseteq \mathcal{S}_n$ and $|\Lambda| \geq 2$.

3 Proof of the Enumeration Results

First, we provide some notations used in proofs. Denote the $\mathcal{C}_n(\Lambda)$ the set of nonnesting permutations that avoid elements in the set Λ. Let $\pi \in \mathcal{C}_n(\Lambda)$. Let $\alpha = \pi_i\pi_{i+1}...\pi_j, \beta = \pi_k\pi_{k+1}...\pi_l$ be two subwords of π such that $j < k$. Denote $\mathrm{Set}(\alpha)$ the set of entries in α without multiplicities. For example, $\mathrm{Set}(113232) = \{1, 2, 3\}$. We say $\alpha < \beta$ if and only if $a < b$ for all $a \in \mathrm{Set}(\alpha)$ and $b \in \mathrm{Set}(\beta)$. This definition holds for other relation, such as $>, =, \leq$, and $\geq$. However, if $\alpha \leq \beta$, then $max\{\mathrm{Set}(\alpha)\} \leq min\{\mathrm{Set}(\beta)\}$. In other words, there is at most one $a \in \alpha$ and one $b \in \beta$ such that $a = b$. Moreover, note that these relations are not transitive or antisymmetric, since the empty word ε trivially satisfies $\varepsilon \leq \alpha$ and $\varepsilon \geq \alpha$ for any α.

$\Lambda \subseteq \mathcal{S}_3$	Formula for $s_n(\Lambda)$	Result in the paper	OEIS Code
$\{123, 321\}$	0, for $n \geq 5$	Theorem 3.3	N/A
$\{123, 231\}$	$\dfrac{(n-1)(n+6)}{2}$, for $n \geq 2$	Theorem 3.5	A055999
$\{132, 213\}$	${F_n}^2$	Theorem 3.6	A007598
$\{132, 231\}$	2^n, for $n \geq 2$	Theorem 3.7	A000079
$\{132, 312\}$	$4 \cdot 3^{n-2}$, for $n \geq 2$	Theorem 3.8	A003946
$\{312, 321\}$		Theorem 3.9	
$\{123, 132, 213\}$	g.f.: $A(x) = \dfrac{1-x}{1-2x-2x^2+2x^3}$	Theorem 3.10	A052528
$\{123, 132, 321\}$	0, for $n \geq 5$	Corollary 3.4	N/A
$\{123, 213, 312\}$	$n+2$, for $n \geq 2$	Theorem 3.11	N/A
$\{132, 213, 312\}$		Theorem 3.12	
$\{132, 213, 321\}$	n, for $n \geq 3$	Theorem 3.13	N/A
$\{132, 312, 321\}$	$4(n-1)$, for $n \geq 2$	Theorem 3.14	N/A
$\{123, 132, 213, 321\}$	0, for $n \geq 5$	Corollary 3.4	N/A
$\{123, 132, 231, 321\}$			
$\{123, 132, 312, 321\}$			
$\{123, 132, 213, 231\}$	4, for $n \geq 2$	Theorem 3.15	N/A
$\{123, 132, 231, 312\}$	2, for $n \geq 3$	Theorem 3.16	N/A
$\{132, 213, 231, 312\}$		Theorem 3.17	
$\{123, 132, 213, 231, 312\}$	1, for $n \geq 3$	Theorem 3.18	N/A
$\{123, 132, 213, 231, 321\}$	0, for $n \geq 5$	Corollary 3.4	N/A

Table 1: A summary of the enumeration of nonnesting permutations avoiding subsets of $\mathcal{S}_3$ of size at least 2.

3.1 Avoiding One Pattern

The enumeration of noncrossing permutations that avoid a single pattern remains open [1]. Complementation and reversal reduces avoidance of a single pattern in nonnesting permutations to the enumeration of $\mathcal{C}_n\{123\}$ and $\mathcal{C}_n\{132\}$, which is also open. The first five entries in the sequence of $s\{123\}$ are $1, 4, 17, 82, 406, 2070$, and the ones in $\mathcal{C}_n\{132\}$ are $1, 4, 17, 77, 367, 1815$. None of these sequences appear in the Online Encyclopedia of Integer Sequences [6] at the time of writing this paper.

Although there is no closed formula for nonnesting permutations that avoid a single pattern of $\mathcal{S}_n$, it is easier to enumerate the ones that avoid a pattern of length 3 with repeated entries. We disregard the trivial case of 111, avoided by all nonnesting permutations. Then, by applying complementation and reversal, we again reduce the problem to enumerating $\mathcal{C}_n\{112\}$ and $\mathcal{C}_n\{121\}$.

Theorem 3.1. $s_n(112) = \mathrm{Cat}_n$.

Proof. View the nonnesting permutations in $\mathcal{C}_n(112)$ as labeled nonnesting matchings. For any $1 \leq i << \leq n$, any subsequence consisting of i and j must be either $jjii$ or $jiji$ to avoid 112. Therefore, the underlying permutations of the nonnesting matchings must be weakly decreasing. It follows that $s_n(112)$ is the number of nonnesting matchings of $[2n]$, known as Cat_n. $\square$

Theorem 3.2. $s_n(121) = n!$.

Proof. For any $1 \leq i << \leq n$, any subsequence consisting of i and j must be either $jjii$ or $iijj$ to avoid 121. Therefore, the permutations in $\mathcal{C}_n(121)$ are obtained by duplicating each entry in a permutation of $\mathcal{S}_n$. Hence, there are $n!$ permutations. $\square$

3.2 Avoiding Two Patterns

Theorem 3.3. *For all $n \geq 5, s_n(123, 321) = 0$.*

Proof. Let $\Lambda = \{123, 321\}$. Since $s_5 = 0$, any permutations of length greater than 5 contains either 123 or 321, or both. $\square$

Corollary 3.4. *For any Λ that contains $\{123, 321\}$, $s_n(\Lambda) = 0$ for all $n \geq 5$.*

Theorem 3.5. *For $n \geq 2$, we have*

$$s_n(123, 231) = \frac{(n-1)(n-6)}{2}$$

.

Proof. Let $\Lambda = \{123, 231\}$. Write $\pi \in \mathcal{C}_n(\Lambda)$ uniquely as $\pi = \alpha 1 \beta 1 \gamma$. Since π avoids 123 and 231, α, β, and γ are weakly decreasing, and we must have $\alpha \geq \beta$ and $\beta \geq \gamma$. Hence, $|\operatorname{Set}(\alpha) \cap \operatorname{Set}(\beta)| \leq 1$ and $|\operatorname{Set}(\beta) \cap \operatorname{Set}(\gamma)| \leq 1$. Avoidance of nesting forces entries in β to be distinct. Therefore, β has at most two entries, leaving us four cases:

(1) $\pi = \alpha 11 \gamma$,

(2) $\pi = \alpha' i 1 i 1 \gamma$ for some $i \in \{2, 3, \ldots, n\}$,

(3) $\pi = \alpha 1 i 1 i \gamma'$ for some $i \in \{2, 3, \ldots, n\}$,

(4) $\pi = \alpha'(i+1)1(i+1)i1i\gamma'$ for some $i \in \{2, 3, \ldots, n-1\}$.

In case (1), avoidance of $\{231\}$ forces the entries of γ to be consecutive, that is, $\operatorname{Set}(\gamma) = \{i+1, i+2 \ldots, j\}$ for some $1 \leq i < j \leq n$. Since α and γ are weakly decreasing, and $\operatorname{Set}(\alpha) = \{2, 3, \ldots, n\} \setminus \operatorname{Set}(\gamma)$, the choice of i and j uniquely determines the permutation. Including the case of $\gamma = \varepsilon$, we have $1 + \binom{n}{2}$ permutations in case (1).

In case (2), the requirement $\beta \geq \gamma$ implies that $\operatorname{Set}(\gamma) = \{2, 3, \ldots, i-1\}$. Since $i \in \{2, 3, \ldots, n\}$, there are $n-1$ permutations in case (2). A similar argument shows that there are $n-1$ permutations in case (3) and $n-2$ in case (4).

Summing up the number of permutations in all cases, we have

$$s_n(\Lambda) = 1 + \binom{n}{2} + (n-1) + (n-1) + (n-2) = \frac{(n-1)(n+6)}{2}.$$

$\square$

Theorem 3.6. *For all $n \geq 1, s_n(132, 213) = {F_n}^2$.*

Proof. Let $\Lambda = \{132, 213\}$. Consider the following four cases of $\pi \in \mathcal{C}_n(\Lambda)$:

(1) $nn\gamma$,

(2) $n\beta n\gamma$,

(3) $\alpha nn\gamma$,

(4) $\alpha n]betan\gamma$,

where $\alpha, \beta \neq \varepsilon$ in cases (2), (3), and (4).

Denote the size of each case as a_n, b_n, c_n, and d_n. Clearly, $s_n(\Lambda) = a_n + b_n + c_n + d_n$. We enumerate $\mathcal{C}_n(\Lambda)$ by considering the ways to generate an elements in $\mathcal{C}_{n+1}(\Lambda)$ by adding two entries $n+1$ to permutations in each case.

First, notice that $n+1$ must be inserted before γ. Otherwise, $nc(n+1)$ would be an occurrence of 213, where $c \in \mathrm{Set}(\gamma)$. Hence, an element in case (1) generates one permutation in each of the four cases: $(n+1)(n+1)nn\gamma$, $(n+1)n(n+1)n\gamma$, $nn(n+1)(n+1)\gamma$, and $n(n+1)n(n+1)\gamma$.

In case (2), the entries $n+1$ must be inserted before β to avoid 213. Hence, each element generates $(n+1)(n+1)n\beta n\gamma$ and $(n+1)n(n+1)\beta n\gamma$, yielding a permutation in case (1) and (2), respectively.

In case (3), to avoid 132, $n+1$ cannot be inserted between the first entry of α and n, leaving us two permutations $(n+1)(n+1)\alpha nn\gamma$ and $\alpha nn(n+1)(n+1)\gamma$ in case (1) and (3), respectively.

The insertion in case (4) faces the restrictions in both cases (2) and (3). So, it only generates $(n+1)(n+1)\alpha n\beta n\gamma$ in case (1).

Adding up the number of permutations generated in each case of $\mathcal{C}_{n+1}(\Lambda)$, we have

$$\begin{aligned}
a_{n+1} &= a_n + b_n + c_n + d_n \\
b_{n+1} &= a_n + b_n \\
c_{n+1} &= a_n + c_n \\
d_{n+1} &= a_n = s_{n-1}(\Lambda).
\end{aligned}$$

Using these relations, we obtain

$$\begin{aligned}
s_{n+1}(\Lambda) &= a_{n+1} + bn + 1 + c_{n+1} + d_{n+1} \\
&= 4a_n + 2b_n + 2c_n + d_n \\
&= 2(a_n + b_n + c_n + d_n) + 2a_n - d_n \\
&= 2a_{n+1} + 2a_n - d_n \\
&= 2s_n(\Lambda) + 2s_{n-1}(\Lambda) - s_{n-2}(\Lambda)
\end{aligned}$$

with the same initial conditions $s_0(\Lambda) = s_1(\Lambda) = 1$ and recurrence as ${F_{n+1}}^2$:

$$\begin{aligned}
{F_{n+1}}^2 &= (F_n + F_{n-1})^2 \\
&= {F_n}^2 + 2F_nF_{n-1} + {F_{n-1}}^2 \\
&= (F_n - F_{n-1})^2 + 4F_nF_{n-1} \\
&= {F_{n-2}}^2 + 4F_nF_{n-1} \\
&= {F_{n-2}}^2 + 2({F_{n+1}}^2 - {F_n}^2 - {F_{n-1}}^2) \\
&= 2{F_n}^2 + 2{F_{n-1}}^2 - {F_{n-2}}^2.
\end{aligned}$$

with the convention $F_0 = F_1 = 1$. $\square$

Theorem 3.7. *For all* $n \geq 2, s_n(132, 231) = 2^n$.

Proof. Let $\Lambda = \{132, 231\}$. The claim clearly holds for $n = 2$ since $s_2(\Lambda) = 4$. Let $n \geq 3$. To generate an element in $\mathcal{C}_n(\Lambda)$ from $\pi \in \mathcal{C}_{n-1}(\Lambda)$, both entries of n must be inserted either at the beginning of end of π to avoid Λ and nestings. Hence,

$$s_n(\Lambda) = 2s_{n-1}(\Lambda).$$

$\square$

Theorem 3.8. *For all* $n \geq 2, s_n(132, 312) = 4(3^{n-2})$.

Proof. Let $\Lambda = \{132, 312\}$ and let $\pi \in s_n(\Lambda)$. Assume that $n \geq 3$. To avoid both 132 and 312, $\pi_{2n} \in \{1, n\}$, leaving us two cases:

(1) $\alpha 1 \beta 1$,

(2) $\alpha n \beta n$.

Since complementation respects avoidance of Λ, it gives a bijection between case (1) and (2). Hence, there are $s_n(\Lambda)/2$ permutations in each case.

In case (1), avoidance of 312 implies that $\alpha \leq \beta$, and the nonnesting condition forces the entries in β to be distinct. Hence, either $\beta = \varepsilon$ or $\beta = n$. If $\beta = \varepsilon$, α can be any permutations in $\mathcal{C}_{n-1}(\Lambda)$, so there are $s_{n-1}(\Lambda)$ such permutations. If $\beta = n$, removing both entries of 1 results in an arbitrary permutation in case (2) of $\mathcal{C}_{n-1}(\Lambda)$. Therefore, $s_n(\Lambda)/2 = s_{n-1}(\Lambda) + s_{n-1}(\Lambda)/2$. With the initial condition $s_2(\Lambda) = 4$, we obtain the result.

$\square$

Theorem 3.9. *For all* $n \geq 2, s_n(312, 321) = 4(3^{n-2})$.

Proof. Let $\Lambda = \{312, 321\}$. Write $\pi \in \mathcal{C}_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$. To avoid both 312 and 321, $|Set(\beta) \cup Set(\gamma)| \leq 1$. This condition, together with the fact that β must have distinct entries, leaves us four cases:

(1) αnn

(2) $\alpha nini$ for some $i \in [n-1]$

(3) $\alpha nnii$ for some $i \in [n-1]$

(4) $\alpha_1 i \alpha_2 nin$ for some $i \in [n-1]$.

Denote the number of permutations in each case as a_n, b_n, c_n, and d_n, so that

$$s_n(\Lambda) = a_n + b_n + c_n + d_n. \tag{1}$$

In case (1), α can be any arbitrary element in $\mathcal{C}_{n-1}(\Lambda)$, so $a_n = sn - 1(\Lambda)$. In case (2), removal of both entries n results in an arbitrary permutation in $\mathcal{C}_{n-1}(\Lambda)$ that ends with a double letter, that is, any permutations from case (1) and (2). Hence, $b_n = a_{n-1} + c_{n-1}$. Similarly, $c_n = a_{n-1} + c_{n-1}$.

In case (4), after removing both entries n, $\alpha_1 i \alpha_2 i$ can be any elements in $\mathcal{C}_{n-1}(\Lambda)$, so $d_n = s_{n-1}(\Lambda)$. Therefore, we can see that $a_n + c_n = b_n + d_n$. Combined with equation 1, we have $s_n(\Lambda) = 2(a_n + c_n)$.

Now we find the recurrence of $s_n(\Lambda)$:

$$\begin{aligned} s_n(\Lambda) &= a_n + b_n + c_n + d_n \\ &= 2s_{n-1}(\Lambda) + 2(a_{n-1} + c_{n-1}) \\ &= 3s_{n-1}(\Lambda) \end{aligned}$$

Given the initial condition $s_2(\Lambda) = 4$, we obtain the results. □

Theorem 3.10. *Let* $\Lambda = \{123, 132, 213\}$. *Then*

$$A(x) = \frac{1-x}{1-2x-2x^2+2x^3}.$$

Proof. Let $\Lambda = \{123, 132, 213\}$. Write $\pi \in \mathcal{C}_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$. To avoid both 123 and 213, $|Set(\alpha) \cup Set(\beta)| \leq 1$. If $|Set(\alpha) \cup Set(\beta)| = 1$, avoidance of 132 forces $\mathrm{Set}(\alpha) \cup \mathrm{Set}(\beta) = n-1$, leaving us four cases:

(1) $nn\gamma$

(2) $(n-1)(n-1)nn\gamma$

(3) $(n-1)n(n-1)n\gamma$

(4) $n(n-1)n\gamma$

In case (1), γ is an arbitrary element in $\mathcal{C}_{n-1}$, and in each of cases (2) and (3), γ can be any permutation in $\mathcal{C}_{n-2}(\Lambda)$. Let D_n represent the set of elements in case (4). Let $|D_n| = d_n$.

In case (4), $(n-1)\gamma$, obtained by removing both entries n, is an arbitrary element in $\mathcal{C}_{n-1}(\Lambda)$ that starts with the largest entry, namely, case (1) and (4) in $\mathcal{C}_{n-1}(\Lambda)$. Since the elements in case (1) of $\mathcal{C}_{n-1}(\Lambda)$ are counted by $s_{n-2}(\Lambda)$, we have

$$d_n = s_{n-2}(\Lambda) + d_{n-1} \tag{2}$$

.

The number of elements in all four cases sums up to

$$s_n(\Lambda) = s_{n-1}(\Lambda) + 2s_{n-2}(\Lambda) + d_n. \tag{3}$$

Solve for d_n in equation 3 and shift the index down by one, we obtain expressions for d_n and d_{n-1}. Substitute the expressions into equation 2 and rearrange it, we obtain

$$s_n(\Lambda) = 2s_{n-1}(\Lambda) + 2s_{n-2}(\Lambda) - 2s_{n-3}(\Lambda)$$

which gives the generating function in Table 1 □

Theorem 3.11. *For all* $n \geq 2$, $s_n(123, 213, 312) = n + 2$.

Proof. Let $\Lambda = \{123, 213, 312\}$. Write $\pi \in s_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$. To avoid both 123 and 213, $|Set(\alpha) \cup Set(\beta)| \leq 1$. To avoid 312, γ is weakly decreasing and $\beta \geq \gamma$. Therefore, we have four cases:

(1) $iinn\gamma$ for some $i \in [n-1]$

(2) $n(n-1)n(n-1)\gamma$

(3) $(n-1)n(n-1)n\gamma$

(4) $nn\gamma$

Since γ is weakly decreasing, there is only one permutation in each of cases (2), (3), and (4), and the choice of i uniquely determines the permutation in case (1). Since there are $n-1$ choices of i in case (1), we have

$$s_n(\Lambda) = (n-1) + 1 + 1 + 1 = n + 2$$

for all $n \geq 2$. □

Theorem 3.12. *For all $n \geq 2$, $s_n(132, 213, 312) = n + 2$.*

Proof. Let $\Lambda = \{132, 213, 312\}$. Write $\pi \in \mathcal{C}_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$. To avoid 213, α and β are weakly increasing, and $\alpha \leq \beta$. Moreover, avoidance of 132 forces $\alpha \geq \beta$, $\beta \geq \gamma$, and $\alpha > \gamma$. Together, these conditions imply two things: (1) $|Set(\beta)| \leq 1$, and (2) $\mathrm{Set}(\alpha) = \mathrm{Set}(\beta) = n-1$ if $\mathrm{Set}(\beta) \neq \emptyset$. Lastly, to avoid 312, γ must be weakly decreasing, leaving us four cases:

(1) $\alpha nn\gamma$

(2) $(n-1)n(n-1)n\gamma$

(3) $n(n-1)n\gamma$.

Since γ is weakly decreasing, there is exactly one permutation in each of the cases (2) and (3). In case (1), the conditions that α weakly increases, γ weakly decreases, and $\alpha > \gamma$ imply that the permutation is uniquely determined by the smallest entry i of α, where $i \in [n]$. Therefore, we have a total of $n + 2$ permutations.

□

Theorem 3.13. *For all $n \neq 2$, we have $s_n(132, 213, 321) = n$.*

Proof. Let $\Lambda = \{132, 213, 321\}$. Write $\pi \in s_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$. To avoid 132, $\alpha \geq \beta$, $\beta \geq \gamma$, and $\alpha > \gamma$. To avoid 213, we must have the words α and β be weakly increasing and $\alpha \leq \beta$. Lastly, avoidance of 321 forces the word γ to weakly increase and $\beta \leq \gamma$.

Suppose $\beta = \varepsilon$. Since α and γ are both weakly increasing, the permutation is uniquely determined by $|\,\mathrm{Set}(\alpha)| \in \{0, 1, \ldots, n-1\}$. Therefore, there are n permutations. Now assume that $\beta \neq \varepsilon$. Then the nonnesting condition requires that either $|\,\mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta)| > 0$ or $|\,\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)| > 0$. In the former case, the conditions $\alpha \leq \beta$ and $\alpha \geq \beta$ imply that the words α and β must be the same set of size 1. However, since $\alpha > \gamma$ and $\alpha = \beta \leq \gamma$, we must have $\gamma = \varepsilon$. Hence, the only permutation that satisfies these requirements is 2121. By the same argument, if β and γ share a common entry, then the permutation must be 1212. Therefore, for $n \neq 2$,

$$s_n = n.$$

□

Theorem 3.14. *For all $n \geq 2$, $s_n(132, 312, 321) = 4(n-1)$.*

Proof. Let $\Lambda = \{132, 312, 321\}$. Write $\pi \in \mathcal{C}_n(\Lambda)$ uniquely as $\pi = \alpha 1 \beta 1 \gamma$. Avoidance of 132 and 321 forces α, β, and γ to all be weakly increasing. To avoid 312, we must have $\alpha \leq \beta$, $\beta \leq \gamma$, and $\alpha < \gamma$, where the inequality is due to the nonnesting condition. Combining these conditions, it follows that $\alpha\beta\gamma = 2233 \cdots nn$, and so $|\operatorname{Set}(\beta)| \leq 2$. Hence, the positions of the 1 entries uniquely determine the permutation. We now consider the potential positions of the 1 entries.

If $\beta = \varepsilon$, the 1 entries are adjacent, and there are n places where they can be. If $|\operatorname{Set}(\beta)| = 1$, the 1 entries are one apart from each other. Since there are $2(n-1)$ entries in the word $\alpha\beta\gamma$, there are $2(n-1)$ permutations. Lastly, if $|\operatorname{Set}(\beta)| = 2$, then we must have $\beta = ij$ for some $2 \leq i < j \leq n$, since there are $n-1$ distinct entries in the word $\alpha\beta\gamma$, there are $(n-1)-1$ permutations.

Summing up all the cases, we have $s_n(\Lambda) = 4(n-1)$.

□

3.3 Avoiding four or five patterns

There are three cases of sets $\Lambda \subseteq \mathcal{S}_3$ of size 4 and one case of size 5 that are not covered by Corollary 3.4. In all of them, the number of nonnesting permutations of $[n] \sqcup [n]$ avoiding Λ is constant for $n \geq 3$.

Theorem 3.15. *For all $n \geq 2$, we have $s_n(123, 132, 213, 231) = 4$.*

Proof. Let $\Lambda = \{123, 132, 213, 231\}$. For $n \geq 3$, any $\pi \in \mathcal{C}_n(\Lambda)$ must be of the form $nn\alpha$, since the avoidance condition requires that in any subsequence $\pi_i\pi_j\pi_k$ of distinct letters, π_i must be the largest. Therefore, $s_n(\Lambda) = s_{n-1}(\Lambda)$ for $n \geq 3$. Since $s_2(\Lambda) = 4$, the result follows. □

Theorem 3.16. *For all $n \geq 3$, we have $s_n(123, 132, 231, 312) = 2$.*

Proof. Let $\Lambda = \{123, 132, 231, 312\}$ and let $n \geq 3$. The avoidance of 132 and 231, together with the nonnesting condition, implies that any $\pi \in \mathcal{C}_n(\Lambda)$ must be of the form αnn or $nn\gamma$. Additionally, avoidance of 123 and 312 forces α and γ must be weakly decreasing. Thus, for $n \geq 3$,

$$\mathcal{C}_n(\Lambda) = \{(n-1)(n-1)(n-2)(n-2)\ldots 11nn, nn(n-1)(n-1)\ldots 11\}. \quad \square$$

Theorem 3.17. *For all $n \geq 3$, we have $s_n(132, 213, 231, 312) = 2$.*

Proof. Let $\Lambda = \{132, 213, 231, 312\}$. Any subsequence of $\pi \in \mathcal{C}_n(\Lambda)$ of length 3 with distinct entries must be increasing or decreasing. Hence, for $n \geq 3$,

$$\mathcal{C}_n(\Lambda) = \{1122\ldots nn, nn\ldots 2211\}. \quad \square$$

Theorem 3.18. *For all $n \geq 3$, we have $s_n(123, 132, 213, 231, 312) = 1$.*

Proof. Any subsequence of $\pi \in \mathcal{C}_n(\{123, 132, 213, 231, 312\})$ of length 3 with distinct entries must be weakly decreasing. Hence, for $n \geq 3$, the only possibility is $\pi = nn\ldots 2211$. □

3.4 Avoiding repeated patterns of length 4

In this section we consider nonnesting permutations avoiding some sets of two or more patterns of length 4, consisting of the symbols $1, 2, 3$.

We start this section with the following lemma:

Lemma 3.19. *For any distinct entries $i,\ j,\ k \in [n]$, $\pi \in \mathcal{C}_n(ijjk)$ if an only if the underlying permutation of π avoids ijk.*

Proof. The reverse direction is trivial. We prove the forward direction. Suppose the underlying permutation of π contains ijk, then it must contain $ijkk$. The second entry of j must occur before the second entry of k to avoid nesting. It follows that π contains $ijjk$. □

Lemma 3.20. *Let π be a nonnesting permutation of $[2n]$, and let $i,\ j,\ k \in [n]$ be distinct entries. If π avoids either $iijk$ or $ijkk$, then π avoids $ijjk$.*

Proof. We prove the contrapositive statement. Let $\pi \in \mathcal{C}_n$ such that π contains $ijjk$. To avoid nesting, the other entry of i must occur before the second entry of j. This creates an occurrence of $iijk$. By symmetry, π also contains $ijkk$. □

The following lemma is a partial converse of lemma 3.20.

Lemma 3.21. *Let $\Lambda = \{iikj, ikkj, ijkj, ikjj, ikjk\}$ for some distinct entries $i,\ j,\ k \in [n]$. For any $\pi \in \mathcal{C}_n(ijjk)$, if π avoids some $\sigma \in \Lambda$, then π avoids $iijk$.*

Proof. Again, we prove the contrapositive statement. Let $\pi \in \mathcal{C}_n(ijjk)$. Suppose π contains $iijk$. Avoidance of $ijjk$ and nestings forces the underlying permutation to be ikj. Therefore, the only two possible subsequences of π are $ikijkj$ and $iikjkj$, both of which contain all patterns in Λ. □

After applying lemma 3.20 and 3.21, we enumerate the following permutations that are not trivially isomorphic. Table 2 lists sets Λ for which we have found a formula for $s_n(\Lambda)$. Other sets for which a formula is conjectured are listed in Table 3.

In the following proofs, it is convenient to consider the permutations in $\mathcal{C}_n$ as nonnesting matchings with each arc labeled with a distinct number in $[n]$. Denote the number of underlying permutations that avoid some set Λ as $|\mathcal{S}_n(\Lambda)| = a_n(\Lambda)$.

Theorem 3.22. $s_n(1223) = \mathrm{Cat}_n^2$.

Proof. We view the nonnesting permuations as labeled nonnesting matchings. By lemma 3.19, avoidance of 1223 indicates that the underlying permutations avoid 123. By MacMahon [5], $a_n(\sigma) = \mathrm{Cat}_n$for any $\sigma \in S_3$. Since there are Cat_n matchings, $s_n(1223) = \mathrm{Cat}_n^2$. □

Theorem 3.23. $s_n(1332) = \mathrm{Cat}_n^2$.

Proof. The argument is the same as the proof of theorem 3.22. □

Theorem 3.24. *For all $n \geq 5$, $s_n(1123, 3211) = 0$.*

Proof. Let $\Lambda = \{1123, 3211\}$ and write $\pi \in \mathcal{C}_n(\Lambda)$ uniquely as $\alpha 1 \beta 1 \gamma$. Avoidance of 1123 and 3221 forces α to be weakly increasing and γ be weakly decreasing. Therefore, $|\,\mathrm{Set}(\alpha)| \leq 2$ and $|\,\mathrm{Set}(\gamma)| \leq 2$. Otherwise, both entries 1 will be part of the occurrence of either pattern. However, if $n \in \mathrm{Set}(\alpha)$, γ must be a word of length 1 or less to avoid 3211. By the same argument, if $n \in \mathrm{Set}(\gamma)$, then $|\,\mathrm{Set}(\alpha)| \leq 1$. It follows that $s_n(\Lambda) = 0$ for all $n \geq 5$. □

Λ	Formula for $s_n(\Lambda)$	Result in the paper	OEIS code
$\{1223\}$	Cat_n^2	Theorem 3.22	A001246
$\{1332\}$		Theorem 3.23	
$\{1123, 3211\}$	0, for $n \geq 5$	Theorem 3.24	NA
$\{1123, 3321\}$		Theorem 3.25	
$\{1123, 1132\}$	$2^{n-1}\,\mathrm{Cat}_n$	Theorem 3.26	A003645
$\{1332, 2113\}$		Theorem 3.27	
$\{1332, 3112\}$		Theorem 3.29	
$\{2113, 3112\}$		Theorem 3.29	
$\{1123, 1132, 2311\}$	$\dfrac{n(n-1)(n+10)}{6}$, for $n \geq 2$	Theorem 3.30	A060488
$\{1132, 2113, 2311\}$	$\dfrac{n^3+6n^2-7n+6}{6}$	Theorem 3.31	A027378
$\{1132, 3112, 3121\}$	$5 \cdot 3^{n-2} - 1$, for $n \geq 2$	Theorem 3.32	A198643
$\{1231, 1321, 2132, 2312, 3123, 3213\}$	$n!F_{n+1}$	Theorem 3.33	A005442

Table 2: A summary of the enumeration of nonnesting permutations avoiding other patterns.

Theorem 3.25. *For all $n \geq 5$, $s_n(1123, 3321) = 0$.*

Proof. The argument is similar to the proof of theorem 3.26. α must be weakly increasing to avoid 3321. It follows that $|\operatorname{Set}(\alpha)| \leq 2$ to avoid 1123. Similarly, γ must be weakly decreasing and $|\operatorname{Set}(\gamma)| \leq 2$. However, if $n \in \operatorname{Set}(\alpha)$, then $|\operatorname{Set}(\gamma)| \leq 1$ to avoid 3321. By the same logic, if $n \in \operatorname{Set}(\gamma)$, then $|\operatorname{Set}(\alpha)| \leq 1$. Therefore, $|\operatorname{Set}(\alpha) \cup \operatorname{Set}(\gamma)| \leq 3$. $\square$

Let σ and τ be a repeated pattern of length 4 that contains 123 and 321 as a subsequence, respectively. Then by lemma 3.20, complementation, and reversal, it follows from theorems 3.24 and 3.25 that $s_n(\sigma, \tau) = 0$ for all $n \geq 5$.

Theorem 3.26. *For all $n \geq 1$, $s_n(1123, 1132) = 2^{n-1}\,\mathrm{Cat}_n$.*

Proof. Given any nonnesting matchings, we consider the possible labelings of the arcs. Avoidance of 1123 and 1132 translates into the condition that no arc has two arcs to its right with larger labels. This is equivalent to counting the underlying permutations that avoid 123 and 132, ie., finding a formula for $a_n(123, 132)$. Therefore, when we label the arcs from left to right, we must choose from the two biggest values available. Since the label of the last arc is forced, there are 2^{n-1} ways to label a nonnesting matching with n arcs. Given that there are Cat_n nonnesting matchings, $s_n(\{1123, 1132\}) = 2^{n-1}\,\mathrm{Cat}_n$. $\square$

Theorem 3.27. $s_n(1332, 2113) = 2^{n-1}\,\mathrm{Cat}_n$.

Proof. By lemme 3.19, this is equivalent to showing $a_n(132, 213) = 2^{n-1}$. Let $\Lambda = \{132, 213\}$. Write the underlying permutation $\pi \in \mathcal{S}_n(\Lambda)$ uniquely as $\alpha n \beta$. To avoid 132, $\alpha > \beta$, and avoidance of 213 forces α to be strictly increasing. Therefore, we can write π as

$$\pi = (i+1)(i+2)\cdots n\beta,$$

where β can be any permutation in $\mathcal{S}_i(\Lambda)$. We can enumerate $\mathcal{S}_n(\Lambda)$ recursively by $a_n(\Lambda) = \sum_{i=0}^{n-1} a_i$, with the initial conditions $a_0 = a_1 = 1$. This is the same recursion as 2^{n-1}. It follows that $s_n(1332, 2113) = a_n\,\mathrm{Cat}_n = 2^{n-1}\,\mathrm{Cat}_n$. $\square$

Theorem 3.28. $s_n(1332, 3112) = 2^{n-1}\,\mathrm{Cat}_n$.

Proof. The argument is similar to the proof above. Let $\Lambda = \{132, 312\}$. Write the underlying permutation $\pi \in \mathcal{S}_n(\Lambda)$ uniquely as $\alpha n\beta$. Avoidance of Λ implies that $\alpha > \beta$ and β is weakly decreasing. Therefore, $a_n(\Lambda) = 2^{n-1}$ and the result follows. $\square$

Theorem 3.29. *For all $n \geq 1$, $s_n(2113, 3112) = 2^{n-1}Cat_n$.*

Proof. Let $\Lambda = \{213, 312\}$. It suffices to show that $a_n(\Lambda) = 2^{n-1}$. First, observe that any underlying permutation must either start or end with the number 1 to avoid both patterns. Therefore, any $\pi \in \mathcal{S}_n(\Lambda)$ must be in the form of either 1α or $\alpha 1$, where α is an arbitrary permutation in $\mathcal{S}_{n-1}(\Lambda)$. It follows that $a_n(\Lambda) = 2a_{n-1}(\Lambda)$. Using the initial condition $a_1(\Lambda) = 1$, we can see that $a_n = 2^{n-1}$. Since there are Cat_n matchings, the result follows. $\square$

In the remaining proofs, we do not view nonnesting permutations as labeled nonnesting matchings.

Theorem 3.30. *For all $n \geq 2$, we have*

$$s_n(1123, 1132, 2311) = \frac{n(n-1)(n+10)}{6}.$$

Proof. Let $\Lambda = \{1123, 1132, 2311\}$. Write $\pi \in \mathcal{C}(\Lambda)$ as $\pi = \alpha 1\beta 1\gamma$. To avoid both 1123 and 1132, we must have $|\operatorname{Set}(\gamma)| \leq 1$.

Moreover, avoidance of 2311 forces α to be weakly decreasing, and the nonnesting condition implies that β has no repeating entry. Therefore, after removing from β the entry in $\operatorname{Set}(\beta) \cap Set(\gamma)$, if there is any, the remaining entries in β must be decreasing. This is because the entries also occur in α, and π is nonnesting. Additionally, the weakly decreasing condition of α implies that the entries that appear in α but not in β must be bigger than those that do. This leaves the following possibilities:

If $\gamma = \varepsilon$, the above conditions imply that

$$\pi = nn(n-1)(n-1)\cdots(i+1)(i+1)i(i-1)\cdots 21i(i-1)\cdots 21,$$

where $i \in [n]$. Hence there are n permutations in this case.

Suppose $\gamma = cc$ for some $c \in \{2, \ldots n\}$. If $\beta = \varepsilon$, each of the $n-1$ choices of c determines the permutation. Now suppose that $\beta \neq \varepsilon$, and let b be the first entry in β. If $c < b$, we need $c = b-1$ to avoid 2311, so there are $n-2$ permutations. If $c > b$, there are no restrictions on the value of c. Since c and b completely determine the permutation, there are $\binom{n-1}{2}$ permutations. Summing up the cases, the number of permutations when $\gamma = cc$ for some c is

$$\frac{n^2+n-4}{2}.$$

Lastly, suppose $\gamma = c$ for some $c \in \{2, \ldots n\}$, so $c \in \operatorname{Set}(\beta)$. If $\beta = c$, the $n-1$ choices of c completely determines the permutation. Otherwise, recall the that entries in β after removing c must be decreasing. Let b be the first of these entries; equivalently, b is the largest element of $\operatorname{Set}(\beta) \setminus \{c\}$. If $b < c$, then there are no restrictions on the value of c and its position in β. Therefore, the values of c and b, and the position of c in β determine the permutation. For each

$c \in \{3, \ldots, n\}$, $b \in \{2, \ldots, c-1\}$, and there are b places where c could be in β. Therefore, the number of permutations in this case is

$$\sum_{2 \leq b < c \leq n} b = \sum_{b=2}^{n-1} b(n-b) = \frac{n^3 - 7n + 6}{6}.$$

On the other hand, if $b > c$, consider the two following subcases. If c is the first entry in β, then any of the $\binom{n-1}{2}$ choices of b and c completely determine the permutation. Otherwise, we need $b = c + 1$ to avoid 2311. Hence, π is given by the value of b and the position of c, of which there are $b - 2$ choices, giving

$$\sum_{b=3}^{n} (b-2) = \frac{(n-1)(n-2)}{2}.$$

By summing up all the cases, we have

$$s_n(\Lambda) = n + \frac{n^2 + n - 4}{2} + n - 1 + \frac{n^3 - 7n + 6}{6} + (n-1)(n-2) = \frac{(n-1)(n)(n+10)}{6}.$$

$\square$

Theorem 3.31. *For all $n \geq 1$, we have*

$$s_n(1132, 2113, 2311) = \frac{n^3 + 6n^2 - 7n + 6}{6}.$$

Proof. Let $\Lambda = \{1132, 2113, 2311\}$. We can write $\pi \in \mathcal{C}_n(\Lambda)$ as $\pi = \alpha 1 \beta 1 \gamma$. To avoid 1132, 2311 and 2113, we must have the word γ be weakly increasing, α be weakly decreasing, and $\alpha > \gamma$, respectively. It follows that, after removing from β the entries in $\mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta)$, if there are any, the remaining entries in β must be strictly increasing, because the entries also appear in γ, and π is nonnesting. It follows that the nonnesting condition forces any entries in γ that do not appear in β to be greater than those that appear in both. By the same argument, the remaining entries in β after removing entries in $\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)$, if any, are strictly decreasing. Additionally, entries in α but not in β are greater than the entries that appear in both. Now, we separate the permutations into three cases:

(1) $|\mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta)| \geq 2$,

(2) $\mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta) = \{b\}$ for some $b \in \{2, 3, \ldots, n\}$,

(3) $\mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta) = \emptyset$.

In case (1), the condition that $\alpha > \gamma$ and avoidance of 2311 force $\mathrm{Set}(\gamma) \subseteq \mathrm{Set}(\beta)$. Moreover, in the word β, any entry in $\mathrm{Set}(\gamma) \cap \mathrm{Set}(\beta)$ must appear to the left of any entry in $\mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta)$. Therefore, if $\mathrm{Set}(\alpha) \setminus \mathrm{Set}(\beta) \neq \emptyset$, the permutation π must be of the form

$$\pi = nn(n-1)(n-1) \cdots (i+1)(i+1)i(i-1) \cdots j123 \cdots (j-1)i(i-1) \cdots j123 \cdots (j-1),$$

where $2 \leq j < i \leq n$ and $j + 1 \leq i$. The second inequality results from the assumption that $|\mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta)| \geq 2$. This leaves us $\binom{n-1}{2} - (n-2)$ permutations. On the other hand, if $\mathrm{Set}(\alpha) \setminus \mathrm{Set}(\beta) = \emptyset$, then the permutation π would be in the form of

$$\pi = n(n-1) \cdots j1n(n-1) \cdots j23 \cdots (j-1)123 \cdots (j-1),$$

for some $j \in \{2, \ldots, n-2\}$. Adding up all the cases, it follows that there are $\binom{n-1}{2}$ permutations in case (1).

In case (2), the nonnesting condition forces b to be the smallest entry in the word α, as discussed above. There are no restrictions on the position of b in β. Let i be the entry in β that precedes b, and let j be the largest entry in β other than b. Hence, $1 \le i \le j < b \le n$. Notice that the case of $\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma) = \emptyset$ occurs when $1 = i = j$. Hence, the $\binom{n+1}{3}$ choices of i, j, and b determine the permutations in case (2).

In case (3), the choices of $i := min\{\mathrm{Set}(\alpha)\}$ and $j := max\{\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)\}$, with $1 \le j < i \le n+1$, uniquely determine the permutation. Here, we say $i = n+1$ if $\alpha = \varepsilon$ and $j = 1$ if $\beta = \varepsilon$. Hence we have $\binom{n+1}{2}$ permutations.

By adding up all the cases, it follows that

$$\binom{n-1}{2} + \binom{n+1}{3} + \binom{n+1}{2} = \frac{n^3 + 6n^2 - 7n + 6}{6}.$$

□

Theorem 3.32. *For all $n \ge 2$, $s_n(1132, 3112, 3121) = 5 \cdot 3^{n-2} - 1$*

Proof. Let $\Lambda = \{1132, 3112, 3121\}$ and write $\pi \in \mathcal{C}_n(\Lambda)$ as $\pi = \alpha 1 \beta 1 \gamma$. To avoid 1132, γ is weakly increasing. Together with the nonnesting condition, we need any entries in $\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)$ be weakly increasing in β. Also, if $c \in \mathrm{Set}(\gamma) \setminus \mathrm{Set}(\beta)$ and $b \in \mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)$ then $c > b$. In order for π to avoid 3112 and 3121, we need $\alpha < \gamma$ and $\alpha \le \beta$, respectively. This implies two restrictions: firstly, if $\mathrm{Set}(\alpha) \setminus \mathrm{Set}(\beta) \ne \emptyset$, then $|\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)| \le 1$ to avoid 1132. Secondly, $|\mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta)| \le 1$. The latter condition leaves us two cases:

(a) $\mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta) = \emptyset$

(b) $\mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta) = \{b\}$ for some $2 \le b \le n$.

In case (a), if $\alpha \ne \varepsilon$, then $|\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)| \le 1$ to avoid 1132, as discussed above. Since γ is weakly increasing and $c > b$ for any $c \in \mathrm{Set}(\gamma) \setminus \mathrm{Set}(\beta)$ and $b \in \mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)$, there are two possibilities $\pi = \alpha 11 \gamma$ and $\pi = \alpha 1b1b\gamma$ for some $3 \le b \le n$. Let i be the length of α. In these two possibilities, $1 \le i \le n-1$ and $1 \le i \le n-2$, respectively, giving us $\sum_{i=1}^{n-1} cc_i(\Lambda) + sum_{i=1}^{n-2} cc_i(\Lambda)$ permutations. If $\alpha = \varepsilon$, $|\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)|$ uniquely determines the permutation. Since this size ranges from 0 to $n-1$, there are $s_{n-1}(\Lambda) + 2\sum_{i=1}^{n-2} cc_i(\Lambda) + n$ permutations in case (a).

In case (b), the requirement $\alpha \le \beta$ forces $b = max\{\mathrm{Set}(\alpha)\}$ and $b < \gamma$. We break it down into two subcases:

(1) $\mathrm{Set}(\alpha) \setminus \mathrm{Set}(\beta) = \emptyset$,

(2) $\mathrm{Set}(\alpha) \setminus \mathrm{Set}(\beta) \ne \emptyset$.

In case (2.1), $\alpha = b$. If $|\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)| \ne 1$, then avoidance of 1132 requires that the last entry of the word β is b. Since $|\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)| \in \{0, 2, 3, \ldots, n-2\}$, there are $n-2$ number of such words. If $|\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)| = 1$, the only two possibilites are $\pi = 2123134455\ldots nn$ and $\pi = 2132134455\ldots nn$. So there are n permutations in case (2.1).

In case (2.2), $|\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)| \le 1$ since π avoids 1132, leaving us two possibilities: $\pi = \alpha 1bc1c\gamma$ and $\alpha 1b1\gamma$, where $\{c\} = \mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)$ if $|\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)| = 1$. Let $i = |\mathrm{Set}(\alpha)|$. In both possibilities, αb is an arbitrary permutation in $\mathcal{C}_i(\Lambda)$ such that the last entry is the largest entry. The three requirements that γ is weakly increasing, $\alpha < \gamma$, and $b < \gamma$ imply that αb uniquely determines the permutation.

Now, rewrite αb as $\alpha b = \alpha\prime 1\prime\beta\prime 1\prime\gamma\prime$, where b is an entry in the word $\gamma\prime$. If $b \notin \mathrm{Set}(\beta\prime)$, then the permutation obtained by removing the entries b is an arbitrary permutation in $\mathcal{C}_{i-1}(\Lambda)$. If $b \in \mathrm{Set}(\beta\prime)$, let g_i enumerate these permutations. Define

$$f_i = s_{i-1}(\Lambda) + g_i. \tag{4}$$

Then, the number of words in case (2.2) is $\sum_{i=1}^{n-2} f_i + \sum_{i=1}^{n-1} f_i$.

Now to obtain a recurrence of g_i, we enumerate $\pi \in \mathcal{C}_n(\Lambda)$ subject to the following conditions: if we rewrite π as $\pi = \alpha 1\beta 1\gamma$, then $\gamma \subseteq \beta$ and the $\pi_{2n} = n$. We separate them into two cases (a) and (b) as above. In case (a) where $\mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta) = \emptyset$, either $\pi = \alpha 1n1n$ for any $\alpha \in \mathcal{C}_{n-2}$ or $\pi = 12...n12...n$. In case (b) where $|\mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta)| = 1$, either $\pi = \alpha 1(n-1)n1n$ for any $\alpha(n-1) \in \mathcal{C}_{n-2}(\Lambda)$, or $\pi = 2134..(n-1)n2134...(n-1)n$. Hence

$$g_n = (s_{n-2} + 1) + (f_{n-2} + 1). \tag{5}$$

Combining equations (4) and (5), we have

$$f_i = s_{i-1}(\Lambda) + s_{i-2}(\Lambda) + f_{i-2} + 2. \tag{6}$$

Summing up all the cases, we have

$$s_n(\Lambda) = n + s_{n-1}(\Lambda) + 2\sum_{i=1}^{n-2} s_i(\Lambda) + f_{n-1} + 2\sum_{i=1}^{n-2} f_i + n. \tag{7}$$

Reduce the index in (7) to obtain an expression of $s_{n-1}(\Lambda)$, then add $s_{n-1}(\Lambda)$ and subtract its associated expression on the right hand side of (7), we simplify to

$$\begin{aligned} s_n(\Lambda) &= 2s_{n-1}(\Lambda) + 2 + s_{n-2}(\Lambda) + f_{n-1} + f_{n-2} \\ &= 2s_{n-1}(\Lambda) + 2 + s_{n-2}(\Lambda) + f_{n-1} + (f_n - s_{n-1}(\Lambda) - s_{n-2}(\Lambda) - 2) \\ &= s_{n-1}(\Lambda) + f_n + f_{n-1}. \end{aligned} \tag{8}$$

Given equations (4) and (8), we can show that $f_n = 2f_{n-1} + 3f_{n-2}$ and $s_n(\Lambda) = 3s_{n-1}(\Lambda) + 2$ using induction. Using the initial condition $s_0(\Lambda) = 1$, $s_1(\Lambda) = 1$, we obtain the result. □

Theorem 3.33. *For all $n \geq 1$, $s_n(1231, 1321, 2132, 2312, 3123, 3213) = n!F_{n+1}$, where F_n is the Fibonacci number.*

Proof. Let $\Lambda = \{1231, 1321, 2132, 2312, 3123, 3213\}$. To avoid Λ, any $\pi \in \mathcal{C}(\Lambda)$ must avoid $ijki$ with distinct entries i, j, k, for some $1 \leq i, j, k \leq n$. Then the associated matchings must end with either a single arc without crossing or exactly two arcs crossing each other. Viewing the nonnesting permutations as nonnesting matchings, we see that the number of matchings has the recurrence $a_n = a_{n-1} + a_{n-2}$. Given the initial conditions $a_1 = 1, a_2 = 2$, it follows that $a_n = F_{n+1}$. Each matching can be labeled in any way to form a permutation in $\mathcal{C}_n(\Lambda)$. Since there are $n!$ possibly labeling, $s_n(\Lambda) = n!F_{n+1}$. □

The table below summarizes the conjecture for other patterns.

Λ	Conjecture for $s_n(\Lambda)$	OEIS code
$\{1322\}$	$\frac{1}{n}\sum_{k=0}^{n}\binom{3n}{n-k-1}\binom{n+k-1}{k}$	A007292
$\{1123, 1322\}$	$\sum_{k=0}^{n}\binom{n+k-1}{n}\binom{k}{n-k}$	A055834
$\{1132, 3112\}$		
$\{1322, 2113\}$		
$\{1132, 2213\}$	$\frac{1}{2x}((1-x)^2-\sqrt{(1-x)^4-4x(1-x)^2})$	A006319
$\{1233, 1322\}$		
$\{1231, 1321\}$	$\sum_{k=0}^{\infty} 2^{n+1}\frac{k^n}{3^{k+1}}$	A122704
$\{1231, 3213\}$	e.g.f.: $\frac{1}{\cos(\log(1-x))+\sin(\log(1-x))}$	A184942
$\{1231, 1321, 2113\}$	$\sum_{k=0}^{n+1} 3^k\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$	A001263
$\{1231, 3213, 3123\}$	e.g.f.: $\frac{1}{3+x-2\exp(x)}$	A292932
$\{1231, 1312, 2231, 3221\}$	g.f.: $\frac{1-3x+2x^2}{(1-3x)(1-x-x^2)}$	A099159
$\{1231, 1213, 2132, 2231\}$	$\frac{11\cdot 4^{n-2}+1}{3}$	A199210
$\{1312, 2132, 2213, 3231\}$	g.f.: $\frac{1}{2x}+x-\frac{\sqrt{1-4x-4x^2+4x^4}}{2x}$	A259845
$\{1123, 2113, 2311\}$	$\lceil 3(n-1)log(3n-3)\rceil$, for $n\geq 2$	A212460

Table 3: Conjectures of the enumeration of nonnesting permutations avoiding other patterns.

4 General Nonnesting Permutation

This section enumerates nonnesting permutations of any multiset. Yan et al. [7] shows that the number of noncrossing permutations of any multiset only depends on the number of distinct elements and the size of the multiset. Using this knowledge, we show the same statement holds for nonnesting permutations.

Consider a general multiset $\mathcal{M}=\{1^{k_1},2^{k_2},\dots,n^{k_n}\}$ with k_i copies of i for each $i=1,2,\dots,n$, where $k_i>0$. Let $K=\sum_{i=1}^{n}k_i$. We denote the set of all permutations of $\mathcal{M}$ by $\mathcal{S}_{\mathcal{M}}$. A permutation $\pi=\pi_1\pi_2\dots\pi_K\ \in\mathcal{S}_{\mathcal{M}}$ is nonnesting if $\nexists$ indices $i<j<k<l$ such that $\pi_i=\pi_l$, $\pi_j=\pi_k$, but $\pi_p\neq\pi_i$ for each $p=j,j+1,\dots,k-1,k$. For a multiset $\mathcal{M}$, denote $\mathcal{C}_{\mathcal{M}}$ the set of all nonnesting permutations of $\mathcal{M}$. A permutation $\pi\ \in\mathcal{S}_{\mathcal{M}}$ is noncrossing if $\nexists$ indices $i<j<k<l$ such that $\pi_i=\pi_k$, $\pi_j=\pi_l$, but $\pi_i\neq\pi_j$. Let $\overline{\mathcal{Q}}_{\mathcal{M}}$ be the set of all noncrossing permutations of $\mathcal{M}$.

Theorem 4.1. *Let $\mathcal{M}=\{1^{k_1},2^{k_2},\dots,n^{k_n}\}$, and let $K_M=k_1+k_2+\dots+k_n$. Then $|\mathcal{C}_{\mathcal{M}}|$ only depends on K and n.*

Proof. Suppose that $k_a>1$ for some positive number $a\leq n$. Let $\mathcal{M}_b$ be the multiset obtained from $\mathcal{M}$ by replacing one element a with b for some $1\leq b\neq a\leq n$. We want to show that $|\mathcal{C}_{\mathcal{M}}|=|\mathcal{C}_{\mathcal{M}_b}|$.

We will prove the theorem by showing that there is a bijection between $\mathcal{C}_{\mathcal{M}}$ and $\overline{\mathcal{Q}}_{\mathcal{M}}$ and between $\overline{\mathcal{Q}}_{\mathcal{M}}$ and $\overline{\mathcal{Q}}_{\mathcal{M}_b}$.

A bijection between noncrossing and nonnesting permutation is deduced from [2]*Theorem 3.1 as follows:

Let π be a nonnesting permutation of $\mathcal{M}$. For each $i=1,2,\dots,n$ replace $\pi_j=i$ with a placeholder if π_j is not the first occurrence of i. Let m_i be the number of times i is replaced by a placeholder. Then replace the first placeholder by the nearest i on its left such that $m_i \geq 1$. Note that m_i decreases by one after each time i replaces a placeholder. Repeat the process until $m_i = 0$ for each i. The following example illustrates this process:

Let $\mathcal{M} = \{1^3, 2^4, 3, 4^3, 5^2, 6^2, 7\}$.

1232244554216761 $\to$ 123__4_5____67__ $\to$ 1232_4_5____67__ $\to$ 123224_5____67__ $\to \ldots \to$ 1232244554216761

Under this construction, there is a unique way to replace the placeholders by a number. For a given noncrossing permutation, to reverse the process, replace any number that is not the first occurrence with a placeholder. Working from left to right, replace the jth placeholder with the jth leftmost number i such that $m_i \geq 1$. For example, let $\mathcal{M} = \{1^4, 2^3, 3, 4^3, 5^2, 6^2, 7\}$. Then 12213145665441 is a noncrossing permutation of $\mathcal{M}$.

1222131456765441 $\to$ 12___3_4567_____ $\to$ 121__3_4567_____ $\to$ 12123_4567_____ $\to \ldots \to$ 1212132456714564

The bijection below between $\overline{\mathcal{Q}}_{\mathcal{M}}$ and $\overline{\mathcal{Q}}_{\mathcal{M}_b}$ is a simplification of the bijective trees in [7]*Theorem 2.3.

Given $\pi \in \overline{\mathcal{Q}}_{\mathcal{M}}$, let i, j, k be the first and the last two indices where $\pi_i = \pi_j = \pi_k = a$, respectively. Let p and q be the first and the last indices where $\pi_p = \pi_q = b$, respectively. Let $B_1 = \pi_p \pi_{p+1} \ldots \pi_q$, $B_2 = \pi_i \pi_{i+1} \ldots \pi_j$, and $B_3 = \pi_{j+1} \pi_{j+2} \ldots \pi_k$. If B_1 is contained in B_3, exchange the positions of B_1 and B_2, and then replace π_k with b. Otherwise, move B_3 to the place right after B_1, and then replace π_k with b. See below for examples, where $a = 1, b = 2$.

$$55\,\underbrace{198991}_{B_2}\,3\,\overbrace{\underbrace{242662}_{B_1}\,3771}^{B_3}\,5 \to 55\,\underbrace{242662}_{B_1}\,3\,\underbrace{198991}_{B_2}\,\overbrace{377}^{B_3\prime}2\,5$$

$$55\,\underbrace{13\,\overbrace{24422}^{B_1}\,3773191}_{B_2}\,\overbrace{866881}^{B_3}\,5 \to 55\,\underbrace{13\,\overbrace{24422}^{B_1}\,\overbrace{866882}^{B_3\prime}\,3773191}_{B_2\prime}\,5$$

One can clearly reverse the procedure. So this map between $\overline{\mathcal{Q}}_{\mathcal{M}}$ and $\overline{\mathcal{Q}}_{\mathcal{M}_b}$ is a bijection.

The bijection between $\mathcal{C}_{\mathcal{M}}$ and $\mathcal{C}_{\mathcal{M}\prime}$ where $\mathcal{M}\prime$ is an arbitrary multiset satisfying $K_{\mathcal{M}} = K_{\mathcal{M}\prime}$ is constructed by repeating the two algorithms described above until the associated multiset of the permtuation is $\mathcal{C}_{\mathcal{M}\prime}$.

This completes the proof. $\square$

Corollary 4.2. *Let $\mathcal{M} = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}$ where $k_i \neq 0$ and $K = \sum_{i=1}^n k_i$. Then $\mathcal{C}_{\mathcal{M}} = (K - n + 2)^{n-1}$.*

Proof. This is a direct result from Theorem 4.1 by considering the multiset $\{1^{K-n+1}, 2, 3, ..., n\}$. $\square$

5 Open Questions

Avoid other patterns. It remains open to enumerate nonnesting permutations that avoid only one pattern of length 3. It may also be interesting to avoid other patterns. While we tried to enumerate patterns of length 4, we could not obtain any recurrence of closed formula for the permutations, even when we avoid more than 10 patterns. Therefore, one may consider avoiding some clever combinations of patterns of length 3 and 4.

Generalization. This paper exclusively focuses on pattern avoidance within nonnesting permutations of the multiset $[n] \sqcup [n]$. Exploring pattern avoidance in nonnesting permutations of a more general multiset could be a captivating avenue for further research.

Direct Proof of Theorem 4.1. The proof of theorem 4.1 utilizes the important property of noncrossing permutations. It may be interesting to construct an alternative proof with a short algorithm without using noncrossing permutations.

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