Ultraproducts and Their Applications: An Introduction

A Thesis

Submitted to the Faculty
in partial fulfillment of the requirements for the
degree of

Bachelor of Arts

in

Mathematics

by

Joseph Miles Chionuma Opulauoho

Advised by Peter Winkler

Dartmouth College
Hanover, New Hampshire

June 2026

Acknowledgements

I would like to express my sincere gratitude to my wonderful advisor, Professor Peter Winkler, for his guidance, encouragement, and insight throughout the development of this thesis. His deep knowledge and enthusiasm for all forms of mathematics plus his willingness and patience to share have made this project possible. Completing this work with him has greatly enhanced my mathematical education.

I am also grateful to the Dartmouth College Mathematics Department for providing a stimulating academic environment and for supporting my growth as a student of mathematics. The courses, seminars, and conversations I have experienced at Dartmouth have played an important role in shaping my interests in logic, model theory, and combinatorics. In particular, I'd like to thank Daniel Rockmore for introducing me too and sparking my interesting in Ultraproducts.

I would also like to thank my friends and classmates who have supported me intellectually, spiritually, and in so many other ways. They have made these 4 years at Dartmouth enjoyable and have always pushed me to pursue and study what interest me.

Finally, I am deeply grateful to my family for their unwavering support, patience, and belief in me throughout my time at Dartmouth. My mother, Grace; my father, Kelii; my grandma, Bettye; my grandpa, Henry; and all my wonderful uncles and aunts have encouraged me and served as my rock every single day.

Abstract

This paper aims to serve as an introductory guide to ultraproducts and ultrapowers as they relate to the Honors Mathematical Logic Course at Dartmouth. Key preliminaries topics covered include, filters, ultrafilters, generalized cartesian products, and a suite of specific ultrafilter properties. Key theorems, built up from this base, include Łoś's Theorem, subsequently the compactness theorems for countable and uncountable languages, and the Lowenheim-Skolem-Theorem. Further topics include, applications of ultraproducts to non-standard analysis and basic applications to Ramsey Theory.

1 Introduction

Ultraproducts are an important model theoretic tool, initially developed in the 1950's by Jerzy Łoś's following up on work in the 1930's by Skolem in non-standard arithmetic. Ultraproducts use the generalized cartesian product with a reduction over an ultrafilter to piece together, a typically infinite number of structures or sets. To quote Jerome Keisler, "It is attractive because it is algebraic in nature, but preserves all properties expressible in first order logic" [6]. This preservation of first order properties between the constituent and resultant structure, is expressed mathematically through Łoś's Theorem, a landmark results which shows that first order properties of Ultrafilter almost constituent structures and the ultraproduct are the same. Functionally, this means that proving first order statements about the ultraproduct is equivalent to proving them for the constituent structures. This transfer principle, makes ultraproducts a powerful tool for many areas of math concerned with structure.

In logic, ultraproducts have famously been used to prove the compactness theorem using algebraic and semantic methods rather than non-syntactic methods. They have been used in the fields of analysis via construction of the hyper-reals, structural Ramsey theory, topology [1], algebra, graph theory, and even social choice theory. They are also entirely interesting in their own right. For example, the internal structure created by taking the ultraproducts and the methods by which their cardinality can be controlled are their own topics of papers [5].

2 Key types of Filters

A **filter** $\mathcal{F}$ on a set X is a subset of $\mathcal{P}(X)$ such that:

1) $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$
2) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$
3) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$

An **ultrafilter** on X is a filter $\mathcal{U}$ such that for all $A \subseteq X$, $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$, but not both.

A **principal filter** generated by an element $s \in X$ is given by $\mathcal{F}_s = \{A \subseteq X : s \in A\}$. A **non-principal** filter is a filter which is not principal.

A **free filter** is a filter $\mathcal{F}$ such that $\bigcap_{A \in \mathcal{F}} A = \emptyset$.

A **maximal filter** is a filter $\mathcal{F}$ such that if $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{F}'$ is a filter, then $\mathcal{F} = \mathcal{F}'$.

3 Ultrafilter Theorems and Key Axioms

In this section, we cover and provide proofs for some of the basic results about ultrafilters that will be needed later. These results explain both how ultrafilters behave as a notion of largeness on a set and why they are available for use in ultraproduct constructions. In particular, the ultrafilter theorem will let us extend ordinary filters to ultrafilters, and this will give non-principal ultrafilters on infinite sets.

Proposition 3.1. *For a nonempty set X, all principal filters on X are ultrafilters.*

Proof. Let $\mathcal{F}$ be a principal filter on X generated by some element $s \in X$. For any set $A \subseteq X$, either $s \in A$ or $s \in X \setminus A$, but not both. Thus for any such A, either A or $X \setminus A$ is in $\mathcal{F}$, but not both, so $\mathcal{F}$ is an ultrafilter. $\square$

Proposition 3.2. *An ultrafilter $\mathcal{U}$ is free if and only if it is non-principal.*

Proof. Suppose an ultrafilter $\mathcal{F}$ is non-principal. Then it does not contain any singleton, so it contains the complement of each singleton. The intersection of all these complements is $\emptyset$, so $\mathcal{F}$ is free. Conversely, if an ultrafilter $\mathcal{F}$ is free, then $\bigcap_{A \in \mathcal{F}} A = \emptyset$, so there is no element contained in every set of $\mathcal{F}$. Hence there is no $s \in X$ such that $\mathcal{F} = \{A \subseteq X : s \in A\}$, and thus $\mathcal{F}$ is non-principal. $\square$

Now that some basic filter properties have been discussed, we turn attention to filter construction. This will prove useful later when constructing non-principal ultrafilters over infinite sets, and will eventually be used in constructing ultraproducts specific properties and specifically will show up in the proof of Compactness. Similar to topological spaces, a filter can be constructed and characterized by a subset called a "filter basis" and can even be constructed from a subset of a filter basis, called a "filter subbasis".

Definition 3.3 (Filter basis)**.** A **filter basis** $\mathcal{B}$ on a set X is a collection of non-empty subsets of X which is closed under finite intersection. That is, if $A, B \in \mathcal{B}$, then there exists $C \in \mathcal{B}$ such that $C \subseteq A \cap B$.

One can construct a filter on X from a filter basis $\mathcal{B}$ by taking all supersets of elements of $\mathcal{B}$. That is, $\mathcal{B}$ generates a filter $\mathcal{F}$ where

$$\mathcal{F} = \{A \in \mathcal{P}(X) : \exists C \in \mathcal{B}, C \subseteq A\}.$$

We now show that the set generated by $\mathcal{B}$ is indeed a filter.

Proposition 3.4. *The set $\mathcal{G}$ generated by a filter basis $\mathcal{B}$ is a filter on X.*

Proof. Let $\mathcal{B}$ be a filter basis, and let $\mathcal{G} = \{A \in \mathcal{P}(X) : \exists B \in \mathcal{B}, B \subset A\}$. First, $X \in \mathcal{G}$, since every $B \in \mathcal{B}$ satisfies $B \subseteq X$. Also, $\varnothing \notin \mathcal{G}$, since every member of $\mathcal{B}$ is nonempty. $\mathcal{G}$ is closed under upwards containment; Consider some $A \in \mathcal{G}$ and some element A' such that $A \subset A'$. We have $B \subset A \subset A'$ so by our construction A' is in $\mathcal{G}$. $\mathcal{G}$ is closed under finite intersection; Consider $A_1, A_2 \in \mathcal{G}$. By our construction there is $B_1, B_2 \in \mathcal{B}$ such that $B_1 \subset A_1$ and $B_2 \subset A_2$ Moreover, $B_1 \cap B_2 \subset A_1 \cap A_2$ and since there is a $C \in \mathcal{B}$ such that $C \subset B_1 \cap B_2$ we have $C \subset A_1 \cap A_2$ and therefore $A_1 \cap A_2 \in \mathcal{G}$ by our construction. $\mathcal{G}$ is a filter. $\square$

Definition 3.5 (Filter basis)**.** A **filter subbasis** $\mathcal{S}$ on a set X is a non-empty collection of subsets of X such that finite intersections are non-empty. That is, for every finite $B \subset \mathcal{S}$, $\bigcap B \neq \varnothing$ The collection of all finite intersections of elements of S forms a basis, so a subbasis generates a filter.

Lemma 3.6. *Let $\mathcal{S} \subseteq \mathcal{P}(X)$ satisfy the finite intersection property. Then the collection*

$$\mathcal{B} = \{\bigcap A : A \subseteq \mathcal{S},\ A \textit{ finite}\}$$

is a filter basis on X.

Proof. Let $\mathcal{S} \subseteq \mathcal{P}(X)$ satisfy the finite intersection property. Define $\mathcal{B} = \{\bigcap A : A \subseteq \mathcal{S},\ A \text{ finite}\}$. We show that $\mathcal{B}$ is a filter basis. Let $B_1, B_2 \in \mathcal{B}$. Then there exist finite sets $A_1, A_2 \subseteq \mathcal{S}$ such that $B_1 = \bigcap A_1$ and $B_2 = \bigcap A_2$. Since $A_1 \cup A_2$ is a finite subset of $\mathcal{S}$, we have $\bigcap(A_1 \cup A_2) \in \mathcal{B}$. Moreover,

$$\bigcap(A_1 \cup A_2) = (\bigcap A_1) \cap (\bigcap A_2) = B_1 \cap B_2.$$

Thus there exists $C \in \mathcal{B}$ such that $C \subseteq B_1 \cap B_2$, and hence $\mathcal{B}$ is closed under finite intersection. Therefore, $\mathcal{B}$ is a filter basis. $\square$

Two bases are said to be equivalent if they generate the same filter, and two subbases are said to be equivalent if they generate the same filter.

Proposition 3.7. *Suppose that $\mathcal{F}$ is a filter on X. Then $\mathcal{F}$ is an ultrafilter if and only if it is a maximal filter.*

Proof. Suppose $\mathcal{F}$ is not maximal. Then there is a filter $\mathcal{F}'$ such that $\mathcal{F} \subset \mathcal{F}'$ and $\mathcal{F} \neq \mathcal{F}'$. Then there exists $A \subseteq X$ such that $A \in \mathcal{F}'$ and $A \notin \mathcal{F}$. Also $X \setminus A \notin \mathcal{F}$, since otherwise $(X \setminus A) \cap A = \emptyset$ would be in $\mathcal{F}$, a contradiction. Therefore $\mathcal{F}$ is not an ultrafilter.

Conversely, suppose that $\mathcal{F}$ is not an ultrafilter. Then there is some $A \subseteq X$ such that $A \notin \mathcal{F}$ and $X \setminus A \notin \mathcal{F}$. Let $\mathcal{B} = \{B \cap A : B \in \mathcal{F}\}$. We claim that $\mathcal{B}$ is a filter basis. If $B \in \mathcal{F}$, then $B \cap A \neq \varnothing$, since otherwise $B \subseteq X \setminus A$, so $X \setminus A \in \mathcal{F}$ by upward closure, contradiction. Also, if $B_1, B_2 \in \mathcal{F}$, then $(B_1 \cap A) \cap (B_2 \cap A) = (B_1 \cap B_2) \cap A$, and $B_1 \cap B_2 \in \mathcal{F}$. Thus $\mathcal{B}$ is a filter basis.

Let $\mathcal{G}$ be the filter generated by $\mathcal{B}$, i.e. $\mathcal{G} = \{C \subseteq X : \text{ for some } B \in \mathcal{F},\ B \cap A \subseteq C\}$. Then $\mathcal{F} \subseteq \mathcal{G}$, since for each $B \in \mathcal{F}$, $B \cap A \subseteq B$. Also $A \in \mathcal{G}$, since $X \in \mathcal{F}$ and $X \cap A = A$. Since $A \notin \mathcal{F}$, we have $\mathcal{F} \subsetneq \mathcal{G}$. Hence $\mathcal{F}$ is not maximal. $\square$

Before proving the ultrafilter theorem, we recall **Zorn's Lemma**: if every chain in a partially ordered set $(P, \leq)$ has an upper bound, then $(P, \leq)$ has a maximal element (this is equivalent to the axiom of choice). We will apply this to the poset of filters on X containing a given filter.

Theorem 3.8 (Ultrafilter Theorem)**.** *Given a filter $\mathcal{F}$ on X, there is an ultrafilter $\mathcal{U}$ on X such that $\mathcal{F} \subseteq \mathcal{U}$.*

Proof. Let $\mathcal{F}$ be a filter on X. Now consider the poset of all filters on X containing $\mathcal{F}$, ordered by inclusion:

$$\mathcal{P} = \{\mathcal{G} \subset \mathcal{P}(X) : \mathcal{G} \text{ is a filter on X and } \mathcal{F} \subset \mathcal{G}\}$$

Consider some arbitrary chain $\mathcal{C}$ in this poset. Now define $\mathcal{G}^* = \bigcup \mathcal{C}$. We show that $\mathcal{G}^*$ is a filter and thus an upper bound on this arbitrary chain.

$\varnothing \notin \mathcal{G}^*$ and $X \in \mathcal{G}^*$ as every element in $\mathcal{C}$ is a filter. If $A, B \in \mathcal{G}^*$ then there are $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{C}$ where $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. As $\mathcal{C}$ is a chain under inclusion either $\mathcal{F}_1 \subset \mathcal{F}_2$ or $\mathcal{F}_2 \subset \mathcal{F}_1$. Call the larger subset $\mathcal{F}_3$. A and B are in $\mathcal{F}_3$ which is filter and hence closed under intersection so $A \cap B \in \mathcal{F}_3$. Therefore, $A \cap B \in \mathcal{G}^*$. Likewise if $A \in \mathcal{G}^*$ and B such that $A \subset B$. A is in some filter $\mathcal{B} \in \mathcal{C}$. And filters are closed upwards by inclusion so $B \in \mathcal{B}$. Hence $B \in \mathcal{G}^*$. Therefore, $\mathcal{G}^*$ is a filter, and an upper bound on the chain $\mathcal{C}$

By Zorn's Lemma, $\mathcal{P}$ has a maximal element $\mathcal{U}$. By Proposition 3.7, $\mathcal{U}$ is an ultrafilter. □

Definition 3.9 (Fréchet filter)**.** Let X be an infinite set. The **Fréchet filter** on X is the collection

$$\mathcal{F} = \{A \subseteq X : X \setminus A \text{ is finite}\}.$$

Corollary 3.10. *The ultrafilter extension of a Fréchet filter on an infinite set X is always free (non-principal).*

Proof. By Theorem 3.8, the Fréchet filter always has an ultrafilter extension. Suppose such an extension $\mathcal{F}$ were principal. Then $\{s\} \in \mathcal{F}$ for some $s \in X$. However, the Fréchet filter contains all cofinite subsets of X, in particular $X \setminus \{s\}$. Hence both $\{s\}$ and $X \setminus \{s\}$ are in $\mathcal{F}$, so their intersection $\emptyset$ is in $\mathcal{F}$, a contradiction. Therefore the ultrafilter extension must be non-principal and hence free. □

From Corollary 3.10, it follows that every infinite set admits a non-principal ultrafilter, since every infinite set has a cofinite subset and thus a Fréchet filter.

Corollary 3.11. *X is finite if and only if every ultrafilter on X is principal*

Proof. Direction 1: Let X be finite, then $\mathcal{P}(X)$ is finite. Let $\mathcal{U}$ be an ultrafilter on X. For all $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$. Assume no singleton of the form $\{s\}$ is in $\mathcal{U}$ where $s \in X$. Then all complements of such singletons relative to X are in $\mathcal{U}$. Taking all such complements and intersecting them, since each omits a distinct element of X, the intersection is $\emptyset$. Hence $\emptyset \in \mathcal{U}$, a contradiction. Therefore, there must be a singleton in $\mathcal{U}$. Moreover, $\mathcal{U}$ is generated by this singleton, since if it contained two distinct singletons, their intersection would be $\emptyset$.

Direction 2: If a set is infinite it is possible to define a free ultrafilter by extending the **Fréchet filter** using theorem 3.8.

□

An ultrafilter may be viewed as a notion of largeness on a set X. The following proposition captures an important feature of this viewpoint: if the union of two sets is large, then at least one of the two sets is already large. This is stronger than what holds for ordinary filters, and it reflects the decisive nature of ultrafilters.

Proposition 3.12. *If $B \cup C \in \mathcal{U}$, where $\mathcal{U}$ is an ultrafilter on X, then $B \in \mathcal{U}$ or $C \in \mathcal{U}$.*

Proof. Suppose $B \cup C \in \mathcal{U}$ and $B \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, $X \setminus B \in \mathcal{U}$. Also $B \cup C \in \mathcal{U}$, so by closure under finite intersection, $(X \setminus B) \cap (B \cup C) \in \mathcal{U}$. But $(X \setminus B) \cap (B \cup C) = C \setminus B \subseteq C$. Since $\mathcal{U}$ is upward closed, it follows that $C \in \mathcal{U}$. Thus $B \in \mathcal{U}$ or $C \in \mathcal{U}$. □

4 Generalized Cartesian Product and Constructing the Ultraproduct on sets and structures

Take an index set I and a collection of sets indexed by I, $\{A_i\}_{i \in I}$.

The Generalized Cartesian Product is defined as

$$\prod_{i \in I} A_i = \left\{ f : I \to \bigcup_{i \in I} A_i \;\middle|\; \forall i \in I,\ f(i) \in A_i \right\}.$$

Take a filter $\mathcal{F}$ on a set I, and a collection $\{A_i\}_i \in I$. We denote the $\mathcal{F} \sim$ relation on $\Pi_{i \in I} A_i$ by declaring $a \sim_{\mathcal{F}} b \iff \{i \in I | a(i) = b(i)\} \in \mathcal{F}$. We claim this is an equivalence relation:

Proof. Take a filter $\mathcal{F}$ on a set I, and a collection $\{A_i\}_i \in I$. Let $a, b, c \in \Pi_{i\in I} A_i$

Reflexive: $\forall i, a(i) = a(i)$ thus $X = \{i|a(i) = a(i)\}$ and $X \in \mathcal{F}$. Therefore $a \sim_F a$

Symmetric: This follows from the symmetry of equality. If $\{i|a(i) = b(i)\} \in \mathcal{F}$ then $\{i|b(i) = a(i)\} \in \mathcal{F}$ as $\{i|a(i) = b(i)\} = \{i|b(i) = a(i)\}$

Transitive: Let $a \sim_{\mathcal{F}} b$ and $b \sim_{\mathcal{F}} c$. Then, $\{i|a(i) = b(i)\} \in \mathcal{F}$ and $\{i|b(i) = c(i)\} \in \mathcal{F}$. $\mathcal{F}$, as a filter is closed under finite intersections. So $\{i|a(i) = b(i)\} \cap \{i|b(i) = c(i)\} \in \mathcal{F}$, which is equivalent to $\{i|a(i) = b(i) = c(i)\} \in \mathcal{F}$. Moreover, $\mathcal{F}$ is closed under subsection and $\{i|a(i) = b(i) = c(i)\} \subseteq \{i|a(i) = c(i)\}$. Therefore, $\{i|a(i) = c(i)\} \in \mathcal{F}$. $a \sim_f c$. $\square$

The collection of these $\mathcal{F}$ equivalence classes on $\{A_i\}_{i\in I}$ is called the **reduced product** and is denoted $\Pi_F A_i$. If $\mathcal{F}$ is an ultrafilter, than this is called the **ultraproduct on sets**.

$$\prod\nolimits_{\mathcal{F}} A_i = \left\{ [f]_{\mathcal{F}} : f \in \prod_{i\in I} A_i \right\}.$$

As we will see, ultraproduct will only be particularly interesting if the ultrafilter chosen for their construction is non-principal, which necessitates by theorem 1.2 that the indexing set be infinite. If the ultrafilter is principal than the division of functions into equivalence classes will be dominated by one element.

Example 1: Let $I = \{1, 2, 3\}$ and for each $i \in I$ let $A_i = \{0, 1\}$. The cartesian product, $\Pi_{i\in I} A_i$ is equal to set of the 8 binary sequences of length 3. Now consider the principal filter on I, generated by 1 and denote said filter as $\mathcal{F}$. To be $\mathcal{F}$ equivalent, $a \sim_{\mathcal{F}} b \iff \{i|a(i) = b(i)\} \in \mathcal{F}$. As an example, consider the sequences $\{1, 1, 1\}$ and $\{0, 1, 1\}$ and denote them f and g respectively. $1 = f(1) \neq g(1) = 0$ so $1 \notin \{i|a(i) = b(i)\} \notin \mathcal{F}$. $f \not\sim_{\mathcal{F}} g$. In particular the $\mathcal{F} \sim$ equivalence relation partitions the set of sequences into classes, dependent on whether the sequence maps 1 to 1 or 1 to 0.

An **ultraproduct** is called an **ultrapower** if for all $i \in I$, $A_i = A$. We denote said ultrapower as $\prod_{\mathcal{F}} A$.

The ultraproduct in this case is an ultraproduct on a set. However, for our extended purposes in logic, we want to make an **ultraproduct on structures** in first order logic.

5 First Order Logic and the Ultraproduct on Structures

The idea behind taking an ultraproduct of structures is that we can create a new (larger) structure from existing structures while preserving the characteristics of the component structures that hold in almost all of them.

To create an **ultraproduct on structures** take a language $\mathfrak{L}$ and a family of structures $\mathcal{M}_i$ indexed by a set I. The definition of an ultraproduct is the same as the previous definition, except now we consider structure's as our component sets. The ultraproduct

$$\prod\nolimits_{\mathcal{F}} \mathcal{M}_i = \left\{ [f]_{\mathcal{F}} : f \in \prod_{i \in I} \mathcal{M}_i \right\}.$$

will be the universe for our new structure. Now that we have a universe for our structure we still need to interpret it by defining functions, constants, and relations on it, and show that all of these are well defined. **Relations** will be defined as an extension of our definition of $\mathcal{F} \sim$ relation, and a relation will be preserved between equivalence classes if it holds almost everywhere.

$$([a_1], [a_2]...a[n]) \in R^{\prod_{\mathcal{F}} \mathcal{M}_i} \iff [i \in I | a_1(i), a_2(i)...a_n(i), \in R^{\mathcal{M}_i}] \in \mathcal{F}$$

where R_i is the relation on $\mathcal{M}_i$. Written in another way:

$$\prod\nolimits_{F} \mathcal{M}_i \models R^{\prod_{\mathcal{F}} \mathcal{M}_i}([a_1], [a_2]...[a_n]) \iff [i \in I | \mathcal{M}_i \models R^{\mathcal{M}_i}(a_1(i), a_2(i), ...a_n(i)] \in \mathcal{F}$$

.

Now we show that relations are well defined.

Proof. Suppose that $a_1 \sim_{\mathcal{F}} b_1$, $a_1 \sim_{\mathcal{F}} b_2$ and $a_1, b_1 \in R$. To prove well R is well defined with respect to our U equivalence classes. We must show that $a_2, b_2 \in R$. Let $A = [i \in I | a_1(i) = a_2(i)]$, $B = [i \in I | b_1(i) = b_2(i)]$, and $C = [i \in I | a_1(i), b_1(i) \in R_i]$. A, B and C are all elements of the ultrafilter $\mathcal{F}$. Therefore, $(A \cap B \cap C) \in \mathcal{F}$.

Rewritten, $\{i \in I | a_1(i) = a_2(i)$ and $b_1(i) = b_2(i)$ and $a_1(i), b_2(i) \in R_i\} \in \mathcal{F}$. $A \cap B \cap C \subseteq [i \in I | a_2(i), b_2(i) \in R_i]$. Therefore, $\{i \in I | a_2(i), b_2(i) \in R_i\} \in \mathcal{F}$ and $a_2, b_2 \in R$. $\square$

This argument easily extends to n-ary relations where $n \in \mathbb{N}$, as the finite intersection of elements in an ultrafilter is still in the ultrafilter.

A function on our new structure, f: $(\prod_F \mathcal{M}_i)^n \to (\prod_F \mathcal{M}_i)$ will be defined as such:

$$f^{\prod_{\mathcal{F}} \mathcal{M}_i}([a_1],[a_2],...[a_n]) = [y] \iff \{i \in I | f^{\mathcal{M}_i}(a_1(i),...,a_n(i)) = y_i\} \in \mathcal{F}$$

Alternatively, one could define a function point wise on individual sequences of equivalence classes and then take the equivalence class of the output to be the output of the function with respect to equivalence classes .

$$y_i = f^{\mathcal{M}_i}(a_1(i),\ldots,a_n(i))$$

for all $i \in I$ and then set

$$[y] = f^{\prod_{\mathcal{F}} \mathcal{M}_i}([a_1],\ldots,[a_n])$$

Now we show that functions are well defined:

Proof. Let $a_n \sim_F b_n$ for all $n \in N$. We must show that $f^{\prod_{\mathcal{F}} \mathcal{M}_i}([a_1],[a_2],...[a_n]) = f^{\prod_{\mathcal{F}} \mathcal{M}_i}([b_1],[b_2],...[b_n])$ which requires showing that their outputs belong to the same equivalence class.

Let $a_n \sim_F b_n$ for all $n \in N$. We must show that $f(a_1,a_2,...a_n) = f(b_1,b_2,...b_n)$ which requires showing that their outputs belong to the same equivalence class.

Let $A = \{i \in I | f^{\mathcal{M}_i}(a_1(i),..a_n(i) = y(i)\}$ and let $B = \{i \in I | f^{\mathcal{M}_i}(a_1(i),..a_n(i) = y(i)\}$. $A,B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F} \Rightarrow f(a_1,a_2,...a_n) \sim_{\mathcal{F}} f(b_1,b_2,...b_n)$. □

Proof that our definitions this is indeed a function:

Proof. Let $f^{\prod_F \mathcal{M}_i}([a_1],[a_2],...[a_n]) = [y_1]$ and $f^{\prod_{\mathcal{F}} \mathcal{M}_i}([a_1],[a_2],...[a_n]) = [y_2]$. Then $f^{\mathcal{M}_i}(a_1(i),...f(a_n(i)) = y_1(i) = y_2(i)$ for all $i \in I$. Therefore $\{i \in I | y_1(i) = y_2(i)\} = I \in \mathcal{F}$, therefore $y_1(i) \sim_{\mathcal{F}} y_2(i)$ and $[y_1] = [y_2]$. □

Theorem 5.1 (Łoś's Theorem). *A first-order formula φ is true in the ultraproduct if and only if the set of indices $i \in I$ such that the formula is true in $\mathcal{M}_i$ is a member of $\mathcal{F}$:*

$$\prod\nolimits_{\mathcal{F}} \mathcal{M}_i \models \varphi \iff \{i \in I \mid \mathcal{M}_i \models \varphi\} \in \mathcal{F}.$$

We prove Łoś Theorem by induction on formulas:

Proof. .

Base step: Atomic formula ϕ is of the form $R^{\prod_{\mathcal{F}} \mathcal{M}_i}([a_i]...[a_n])$ or $[a_1] = [a_2]$ where $[a_n] \in \prod_{\mathcal{F}} \mathcal{M}_i$. ϕ is of the form $R^{\prod_{\mathcal{F}} \mathcal{M}_i}([a_i]...[a_n])$ hold automatically by their definition:

$$\prod_{\mathcal{F}} \mathcal{M}_i \models R^{\prod_{\mathcal{F}} \mathcal{M}_i}([a_1],[a_2]...[a_n]) \iff [i \in I | \mathcal{M}_i \models R^{\mathcal{M}_i}(a_1(i), a_2(i), ...a_n(i)] \in \mathcal{F}$$

Moreover $\prod_{\mathcal{F}} \mathcal{M}_i \models [a_1] = [a_2]$ iff $a_1 \sim_f a_2$. Therefore, $\{i \in I | a_1(i) = a_2(i)\} \in \mathcal{F}$, which is equivalent to $\{i \in I | M_i \models a_1 = a_2\} \in \mathcal{F}$.

Inductive step: Assume Łoś Theorem holds for ϕ and τ

Negation: Let $\psi = \neg\phi$:
$\prod_F \mathcal{M}_i \models \psi$
$\iff \prod_F \mathcal{M}_i \not\models \phi$
$\iff \{i \in I | \mathcal{M}_i \models \phi\} \notin \mathcal{F}$
$\iff \{i \in I | \mathcal{M}_i \models \neg\phi\} \in \mathcal{F}$
$\iff \{i \in I | \mathcal{M}_i \models \psi\} \in \mathcal{F}$

Disjunction: Let $\psi = \phi \vee \tau$:
$\prod_F \mathcal{M}_i \models \psi$
$\iff \prod_F \mathcal{M}_i \models \phi$ or $\prod_F \mathcal{M}_i \models \tau$
$\iff \{i \in I | \mathcal{M}_i \models \phi\} \in \mathcal{F}$ or $\{i \in I | \mathcal{M}_i \models \tau\} \in \mathcal{F}$
$\iff \{i \in I | \mathcal{M}_i \models \phi\} \cup \{i \in I | \mathcal{M}_i \models \tau\} \in \mathcal{F}$
$\iff \{i \in I | \mathcal{M}_i \models \phi$ or $\mathcal{M}_i \models \tau\} \in \mathcal{F}$ by corollary 1.4
$\iff \{i \in I | \mathcal{M}_i \models \phi \vee \tau\} \in \mathcal{F}$
$\iff \{i \in I | \mathcal{M}_i \models \psi\} \in \mathcal{F}$

For clarification on the backwards direction of the third equivalence: Let $A \cup B \in F$. Assume A and $B \notin \mathcal{F}$ then A^c and $B^c \in \mathcal{F}$. Then $A^c \cap B^c \in \mathcal{F}$. But $A^c \cap B^c = (A \cup B)^c \in \mathcal{F}$ by Demorgan's law and $(A \cup B)^c \cap (A \cup B) = \varnothing \notin \mathcal{F}$ which is a contradiction as filters are closed under finite intersection.

Conjunction: Let $\psi = \phi \wedge \tau$:
$\prod_F \mathcal{M}_i \models \psi$
$\iff \prod_F \mathcal{M}_i \models \phi$ and $\prod_F \mathcal{M}_i \models \tau$
$\iff \{i \in I | \mathcal{M}_i \models \phi\} \in \mathcal{F}$ and $\{i \in I | \mathcal{M}_i \models \tau\} \in F$
$\iff \{i \in I | \mathcal{M}_i \models \phi\} \cap \{i \in I | \mathcal{M}_i \models \tau\} \in \mathcal{F}$
$\iff \{i \in I | \mathcal{M}_i \models \phi$ and $\mathcal{M}_i \models \tau\} \in \mathcal{F}$
$\iff \{i \in I | \mathcal{M}_i \models \phi \wedge \tau\} \in \mathcal{F}$

$\iff \{i \in I | \mathcal{M}_i \models \psi\} \in \mathcal{F}$

For clarification on the backwards direction of the third equivalence: Let $A = B \cap C \in \mathcal{F}$ then B and $C \in F$. Let $B \notin \mathcal{F}$ then by definition of ultrafilter $C^c \in \mathcal{F}$. $\forall x \in A$, $x \in B$ and $x \in C$ therefore $x \notin C^c$. But $C^c \in \mathcal{F}$ and $A \in \mathcal{F}$ however their intersection is $\varnothing$ which would also be in $\mathcal{F}$ creating a contradiction. Therefore C must be in the ultrafilter. By symmetry B must also be in the ultrafilter.

Existential Quantifier: Let $\psi = \exists x \phi$

$\prod_F \mathcal{M}_i \models \psi$
$\iff \prod_F \mathcal{M}_i \models \exists x \phi$
$\iff \prod_F \mathcal{M}_i \models \phi([a])$ for some $[a] \in \prod_{\mathcal{F}} \mathcal{M}_i$
$\iff \{i \in I | \mathcal{M}_i \models \phi(a(i))\} \in \mathcal{F}$ by inductive hypothesis
$\iff \{i \in I | \mathcal{M}_i \models \exists x \phi\} \in \mathcal{F}$
$\iff \{i \in I | \mathcal{M}_i \models \psi\} \in \mathcal{F}$ ☐

Now that we have proved the powerful Łoś theorem we can prove on of the most central theorems of FOL, the compactness theorem. First however, we will show an important corollary.

A structure S is said to be **elementarily equivalent** to another structure G if they satisfy the same first order sentences. This is written as $S \equiv G$.

Corollary 5.2. *Let $\prod_{\mathcal{F}} \mathcal{M}$ denote the ultrapower of a structure $\mathcal{M}$ constructed using a non-principal ultrafilter $\mathcal{F}$. Then $\prod_{\mathcal{F}} \mathcal{M}$ is elementarily equivalent to $\mathcal{M}$.*

Proof. Every component structure of the ultraproduct is $\mathcal{M}$ thus for any sentence ϕ for all $i \in I$ either $\mathcal{M} \models \phi$ or $\mathcal{M} \not\models \phi$ (in the exclusive sense of or). If $\mathcal{M} \models \phi$ then $\{i \in I | \mathcal{M}_i \models \phi\} = I \in F$ and if $\mathcal{M} \not\models \phi$ then $\{i \in I | \mathcal{M}_i \models \phi\} = \varnothing \notin F$. Therefore by Łoś Theorem, $\prod_{\mathcal{F}} \mathcal{M}$ is elementarily equivalent to $\mathcal{M}$. ☐

Theorem 5.3 (Compactness Theorem (for countable languages))**.** *A set of first-order sentences has a model if and only if every finite subset of it has a model.*

Proof. Let Σ be a set of first order sentences in a countable language $\mathfrak{L}$. Assume that every finite subset $\Psi \subset \Sigma$ has a first order model, that is there is a $\mathcal{M} \models \Psi$. Given that a countable FOL language only has a countably infinite amount of possible sentences, it is possible to index the sentences of Σ by $\mathbb{N}$. $\Sigma = \{\phi_1, \phi_2 .....\}$.

Now we can specify a collection A of subsets of Σ, where $A = \{\Psi_1, \Psi_2, ...\}$ and $\Psi_1 = \{\phi_1\}$, $\Psi_2 = \{\phi_1, \phi_2\}$, ..., $\Psi_n = \{\phi_1, \phi_2, ...\phi_n\}$.

Given the assumption, that every finite subset of Σ has a model, every Ψ_n has a corresponding model $\mathcal{M}_n$. Take this collection B of models and take the ultraproduct using some non-principal ultrafilter $\mathcal{F}$ on the indexing set $\mathbb{N}$. Said ultraproduct models Σ as for any sentence ϕ_n, ϕ_n is not modeled by at most n structures. That is the number of component models of ϕ_n is cofinite, so the set of indices of those models is a member of the ultrafilter, and the sentence is true in the ultraproduct.

$\square$

Theorem 5.4 (Compactness Theorem (for uncountable languages))**.** *A set of first-order sentences has a model if and only if every finite subset of it has a model. [3]*

Proof. Assume every $i \subset \Sigma$ where i is finite has a model $\mathcal{M}_i$. Let I denote the collection of all finite subsets. Let $\prod \mathcal{M}_i$ be the direct product of the structures. For each $i \in I$, let $A_i = \{j \in I | i \subseteq j\}$. The collection $\mathcal{A}$ of A_i's generates a filter basis, which by theorem 3.8, the ultrafilter theorem, is contained within an ultrafilter $\mathcal{F}$. For any $\phi \in \Sigma$, $A_{\{\phi\}} \in \mathcal{F}$. Likewise, whenever $j \in A_{\{\phi\}}$, $\phi \in j$, therefore $M_j \models \phi$. The set of all j with the property that ϕ holds in M_j is a super-set of A_ϕ, hence it is also in $\mathcal{F}$. Therefore if $\phi \in \Sigma$ then $\prod_{\mathcal{F}} \mathcal{M}_i \models \phi$.

The collection $\mathcal{A}$ of A_i's is a filter basis. Let $A_i, A_g \in \mathcal{A}$ then $i, g \in I$ and i,g are finite and the union of two finite sets is finite so $i \cup g$ is finite and in I. $A_{i \cup g} = \{j \in I | j \subset i \text{ and } j \subset g\}$. Therefore, $A_{i \cup g} \subseteq A_i \cap A_g$. Additionally $\mathcal{A}$ does contain the empty set as any $i \in I$ can be extended by adding some sentence $\phi \in \Sigma / i$, so for all $i \in I$, $A_i \neq \varnothing$. $\square$

6 Ultraproduct and Ultrapower Cardinality

Proposition 6.1. *If $\mathcal{M}$ is a structure with $|\mathcal{M}| = n$ for some $n \in \mathbb{N}$, then $|\prod_{\mathcal{F}} \mathcal{M}| = n$.*

Proof. From Łoś theorem, it follows that the cardinality of a finite structure will be the same as its ultrapower, generated by a non-principal ultrafilter because having a finite cardinality n, is a first order property. $\square$

This simple conclusion naturally encourages us to investigate the cardinality of ultrapowers of infinite structure and the cardinality ultraproducts. The methods of investigation require new resources to answer. As a start to our investigation of ultrapower cardinality I'd like to first show two intuitive bounds.

Proposition 6.2. *An ultraproduct $\prod_{\mathcal{F}} \mathcal{M}$ has cardinality greater than or equal to $\mathcal{M}$. That is, $|\prod_{\mathcal{F}} \mathcal{M}| \geq |M|$.*

This follows from the fact that one can make an injunction of M into $\prod_{\mathcal{F}} \mathcal{M}$ with the function $f(x) = [g_x]$, where $g_x(i) = x$ for all i. If $x \neq y$ then $f(x) = f(y)$ as g_x and g_y differ at all i hence they are not $\mathcal{F}$ equivalent.

Proposition 6.3. *Let M be an L-structure, let I be an index set, and let U be an ultrafilter on I. Then the diagonal map*

$$d : M \to \prod_U M, \qquad d(a) = [c_a],$$

where $c_a(i) = a$ for all $i \in I$, is an elementary embedding. Consequently,

$$|M| \leq \left| \prod_U M \right| \leq |M|^{|I|}.$$

What Proposition 6.3 tells us is that an upper bound on the ultrapower may rely on the indexing set of choice, and by choosing a larger indexing set, we may be able to increase the upper bound. More sophisticated bounds on ultrapowers can also be determined through specific ultrafilters on those indexing sets, which quotient out the direct product in particular ways.

In the remainder of this section, we will survey some of the classical properties of ultrafilters that can be used to control ultraproduct and ultrapower cardinality. With that and Łoś's Theorem, this provides a natural way to create elementary extensions, and from there a version of the Löwenheim–Skolem.

Definition 6.4 (κ-completeness)**.** An ultrafilter $\mathcal{U}$ is called **κ-complete** if whenever $S \subseteq \mathcal{U}$ with $|S| < \kappa$, then $\bigcap S \in \mathcal{U}$.

Definition 6.5 (κ-regular ultrafilter $\mathcal{U}$ and κ-regularizing set)**.** An ultrafilter is said to be κ**-regular** if there is $E \subseteq \mathcal{U}$, $|E| = \kappa$ such that each element of I belongs to only finitely many elements of E. Such a set E is called a κ**-regularizing set** for $\mathcal{U}$.

Definition 6.6 (Regular ultrafilter)**.** An ultrafilter $\mathcal{U}$ on an index set I is called **regular** if it is $|I|$ regular.

Proposition 6.7. *Let $\mathcal{U}$ be an ultrafilter on an index set I with $|I| = \kappa$. Then $\mathcal{U}$ is κ^+-complete if and only if it is principal. Hence, an ultrafilter on a countably infinite set is countably incomplete if and only if it is non-principal. See [4] for more details.*

Proof. Let $\mathcal{U}$ be an principal ultrafilter on I with $|I| = \kappa$. As $\mathcal{U}$ is principal every element in $\mathcal{U}$ contains some element α and indeed for every set $A \subset I$ such that $\alpha \in A$ we have $A \in \mathcal{U}$. Therefore for any subset $S \subset \mathcal{U}$ where $|S| < \kappa^+$, we have $\forall X \in S$, $\alpha \in X$. Therefore, $\alpha \in \bigcap S$. Moreover, $\bigcap S \subset I$, so $\bigcap S \in \mathcal{U}$.

Conversely, suppose that $\mathcal{U}$ is κ^+-complete. Now assume that $\mathcal{U}$ is non-principal. Then for every $\alpha \in I$, we have $\{\alpha\} \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, it follows that $I \setminus \{\alpha\} \in \mathcal{U}$ for every $\alpha \in I$. But the family $\{I \setminus \{\alpha\} : \alpha \in I\}$ has cardinality $\kappa < \kappa^+$, so by κ^+-completeness,

$$\bigcap_{\alpha \in I} (I \setminus \{\alpha\}) \in \mathcal{U}.$$

However, $\bigcap_{\alpha \in I}(I \setminus \{\alpha\}) = \emptyset$, contradicting the fact that $\emptyset \notin \mathcal{U}$. Therefore $\mathcal{U}$ must be principal. □

Lemma 6.8 (Keisler)**.** *Let $\mathcal{F}$ be a countably incomplete ultrafilter, and let $\{A_i\}_{i \in I}$ be a collection of infinite structures. Then $|\prod_{\mathcal{F}} A_i|^{\aleph_0} = |\prod_{\mathcal{F}} A_i|$. See [5] for more details.*

Corollary 6.9. *Assuming the Continuum Hypothesis, if $\mathcal{M}$ is a structure with $|\mathcal{M}| = \aleph_0$ and $\mathcal{F}$ is a non-principal ultrafilter on $\mathbb{N}$, then $|\prod_{\mathcal{F}} \mathcal{M}| = \aleph_1$.*

Proof Method 1. From Proposition 6.7 a non-principal ultrafilter over a countable infinite set is incomplete, so a non-principal ultrafilter over $\mathbb{N}$ is incomplete. Moreover, as $|\mathcal{M}| = \aleph_0$ for every i, $|M_i| = \aleph_0$. Therefore by Lemma 6.8, $(\prod_{\mathcal{F}} \mathcal{M}_i)^{\aleph_0} = (\prod_{\mathcal{F}} \mathcal{M}_i)$. $|\prod_{\mathcal{F}} \mathcal{M}_i|$ is bounded below by

$|M_i|$ thus $|\prod_{\mathcal{F}} \mathcal{M}_i| \geq \aleph_0$. Suppose $|\prod_{\mathcal{F}} \mathcal{M}_i| = \aleph_0$. Then by, Lemma 6.8 $|\prod_{\mathcal{F}} \mathcal{M}_i|^{\aleph_0} = \aleph_0^{\aleph_0} = \aleph_0$ which is false. Therefore $|\prod_{\mathcal{F}} \mathcal{M}_i| \neq \aleph_0$ so $|\prod_{\mathcal{F}} \mathcal{M}_i| > \aleph_0$, which with CH implies, $|\prod_{\mathcal{F}} \mathcal{M}_i| \geq \aleph_1$. Likewise $|\prod_{\mathcal{F}} \mathcal{M}_i| \leq \aleph_1$ as $|\prod_{\mathcal{F}} \mathcal{M}_i| \leq \Pi \mathcal{M}_i = \aleph_0^{\aleph_0} = 2^{\aleph_0} = \aleph_1$. Therefore, $|\prod_{\mathcal{F}} \mathcal{M}_i| = \aleph_1$.

□

Proof Method 2. Assume that it was possible to put every element in $\prod_{\mathcal{F}} \mathcal{M}_i$ into a 1 to 1 correspondence with $\mathbb{N}$. Then we could enumerate them as $S = \{S_1, S_2, S_3, ...\}$. Then we could enumerate a representative of each equivalence class as $S* = \{s_1, s_2, s_3, ...\}$ where s_i is a representation of S_i.

Take $g(i) = 1 + max_{j \leq 1}\{s_i\}_i$. $g(i)$ only matches any s_i for a finite number of times hence it is not equivalent to any of them. Hence, the equivalence class $G \neq S_i$ for any $S_i \in S$. This shows that $|\prod_{\mathcal{F}} \mathcal{M}_i| > \aleph_0$. Now given that $|\Pi \mathcal{M}_i| = |I|^{|M|} = \aleph_0^{\aleph_0} = 2^{\aleph_0}$. It follows that $|\prod_{\mathcal{F}} \mathcal{M}_i| \leq \aleph_1$ as $|\prod_{\mathcal{F}} \mathcal{M}_i| \leq \Pi \mathcal{M}_i$ as it is a reduction of the latter. Hence, $\aleph_0 < |\prod_{\mathcal{F}} \mathcal{M}_i| \leq \aleph_1$. By application of the continuum hypothesis, $|\prod_{\mathcal{F}} \mathcal{M}_i| = \aleph_1$. □

Proposition 6.10. *For every infinite cardinal κ, there exists a regular ultrafilter on an index set of cardinality κ. See [4, Proposition 8.3.8]*

Proposition 6.11. *Let $\mathcal{U}$ be a regular ultrafilter on an index set I with $|I| = \kappa$, and let $\mathcal{M}$ be an infinite structure. Then $|\prod_{\mathcal{U}} \mathcal{M}| = |\mathcal{M}|^{\kappa}$. See [4, Theorem 8.3.9].*

Proof. First, by proposition 6.3 we have

$$\left|\prod_{\mathcal{U}} \mathcal{M}\right| \leq |M|^{|I|} = |M|^{\kappa}.$$

We must show the reverse inequality. Since $\mathcal{U}$ is regular, there is a κ-regularizing family $E \subseteq \mathcal{U}$ with $|E| = \kappa$ such that every $i \in I$ belongs to only finitely many members of E. By the well-ordering theorem, which is equivalent to the axiom of choice, we can fix a well-ordering of E. Let $M^{<\omega}$ denote the set of all finite tuples of M. Since M is infinite, $|M^{<\omega}| = |M|$. Hence there is a bijection $b : M^{<\omega} \to M$, which induces a bijection $\prod_{\mathcal{U}} M^{<\omega} \cong \prod_{\mathcal{U}} \mathcal{M}$. Therefore it is enough to define an injection

$$M^E \hookrightarrow \prod_{\mathcal{U}} M^{<\omega}.$$

to prove the reverse inequality, as from $|E| = \kappa$ we have

$$|M|^\kappa = |M|^{|E|} \leq \left| \prod_{\mathcal{U}} M^{<\omega} \right| = \left| \prod_{\mathcal{U}} \mathcal{M} \right|.$$

Now we must define the injection. For each function $g : E \to M$, define $g' : I \to M^{<\omega}$ as follows. For each $i \in I$, let

$$E_i := \{\alpha \in E : i \in \alpha\}.$$

By regularity, for all $i \in I$, E_i is finite. List its elements in the fixed order as $\alpha_1, \ldots, \alpha_r$, and set

$$g'(i) := (g(\alpha_1), \ldots, g(\alpha_r)).$$

Now define $\rho : M^E \hookrightarrow \prod_{\mathcal{U}} M^{<\omega}$ by $\rho(g) := [g']_{\mathcal{U}}$. We claim that ρ is injective. Suppose $g_0, g_1 : E \to M$ are distinct. Then there is some $\alpha \in E$ such that $g_0(\alpha) \neq g_1(\alpha)$. For every $i \in \alpha$, the finite list E_i contains α, and because we use the same fixed ordering of E, the entries corresponding to α occur in the same position in both $g_0'(i)$ and $g_1'(i)$. Thus $g_0'(i) \neq g_1'(i)$ for every $i \in \alpha$. Since $\alpha \in \mathcal{U}$, we have

$$\{i \in I : g_0'(i) \neq g_1'(i)\} \in \mathcal{U}.$$

Therefore $[g_0']_{\mathcal{U}} \neq [g_1']_{\mathcal{U}}$. Hence ρ is injective. Thus, $|M|^\kappa \leq |\prod_{\mathcal{U}} \mathcal{M}|$, and by combination with the upper bound we have $|\prod_{\mathcal{U}} \mathcal{M}| = |M|^\kappa$.

□

Proposition 6.12 (Elementary Chain Theorem). *If $(M_\alpha)_{\alpha<\lambda}$ is an increasing elementary chain of L-structures and $M_\lambda := \bigcup_{\alpha<\lambda} M_\alpha$, then M_λ is naturally an L-structure and $M_\alpha \preceq M_\lambda$ for every $\alpha < \lambda$ See [4, Theorem A.3.11].*

Theorem 6.13 (Löwenheim–Skolem under GCH for countable languages). *Assume GCH. Let M be an infinite structure in a countable language L, and suppose $|M| = \aleph_0$. Then for every infinite cardinal $\kappa \geq \aleph_0$, there is an elementary extension N of M such that $|N| = \kappa$.*

Proof. We construct by transfinite induction on a family $(M_\alpha)_{\alpha\in\mathrm{Ord}}$ such that $M_0 = M$, $M_\alpha \preccurlyeq M_\beta$ for $\alpha < \beta$, and $|M_\alpha| = \aleph_\alpha$ for every ordinal α.

Base Case: Set $M_0 := M$. Since $|M_0| = \aleph_0$, Corollary 5.4 gives a non-principal ultrapower $M_1 := \prod_{U_0} M_0$ with $|M_1| = \aleph_1$. By Proposition 6.3, $M_0 \preccurlyeq M_1$.

Successor Case: Now suppose M_β has been constructed with $|M_\beta| = \aleph_\beta$. If $\beta + 1$ is a successor stage, let $\kappa := |M_\beta| = \aleph_\beta$. By Proposition 6.10, there is a regular ultrafilter U_β on an index set of cardinality κ. Define $M_{\beta+1} := \prod_{U_\beta} M_\beta$. By Proposition 6.3, $M_\beta \preccurlyeq M_{\beta+1}$. By Proposition 6.11, $|M_{\beta+1}| = |M_\beta|^\kappa = \kappa^\kappa$. Since κ is infinite, $\kappa^\kappa = 2^\kappa$, and by GCH, $2^\kappa = \kappa^+$. Therefore $|M_{\beta+1}| = \kappa^+ = \aleph_{\beta+1}$.

Limit Case: If λ is a limit ordinal, define $M_\lambda := \bigcup_{\beta<\lambda} M_\beta$. By Proposition 6.12, each $M_\beta \preccurlyeq M_\lambda$. Also, $|M_\lambda| = \sup_{\beta<\lambda} |M_\beta| = \sup_{\beta<\lambda} \aleph_\beta = \aleph_\lambda$.

Thus by transfinite induction, for every ordinal α there is an elementary extension M_α of M with $|M_\alpha| = \aleph_\alpha$. Since every infinite cardinal is of the form $\aleph_\alpha$, the result follows. □

7 Linear Orders and the Ultrapower of ω

In searching for applications of ultraproducts, one natural place to begin is with ordered structures. In particular, the ultrapower of ω gives a first example of a nonstandard model: it preserves the first-order theory of $(\mathbb{N}, <)$ while at the same time containing genuinely new elements and new order-theoretic behavior. This will serve as a bridge to the hypernaturals and hyperreals in the next section.

7.1 Linear orders, ω, and order types

Definition 7.1 (Linear order)**.** A **linear order** is a set equipped with a relation $<$ satisfying the following first-order properties:

Irreflexivity: $\forall x \, \neg(x < x)$
Transitivity: $\forall x \forall y \forall z \, ((x < y \wedge y < z) \Rightarrow x < z)$
Trichotomy: $\forall x \forall y \, (x < y \vee y < x \vee x = y)$

We write ω for the order type of $(\mathbb{N}, <)$. Thus ω is not the set $\mathbb{N}$ itself, but rather the ordering pattern carried by $\mathbb{N}$: $0 < 1 < 2 < 3 < \cdots$.

We write ω^* for the reverse order type $\cdots < 3 < 2 < 1 < 0$, and $\omega^* + \omega$ for the order type obtained by placing a copy of ω to the right of a copy of ω^*.

More generally, if A and B are linearly ordered sets with order types α and β, then $\alpha + \beta$ denotes the order type obtained by placing a copy of B after a copy of A. Likewise, $\alpha \cdot \beta$ denotes the order type obtained by taking β many consecutive copies of α.

The basic examples relevant here are:
$\omega =$ order type of $(\mathbb{N}, <)$
$\omega^* + \omega =$ order type of $(\mathbb{Z}, <)$
$(\mathbb{Q}, <)$ is a countable dense linear order with no endpoints
$(\mathbb{R}, <)$ is a dense linear order with no endpoints, but unlike $(\mathbb{Q}, <)$ it is complete in the sense that every nonempty set bounded above has a least upper bound

Thus $(\mathbb{N}, <)$ is discrete and has a least element, $(\mathbb{Z}, <)$ is discrete with neither least nor greatest element, while $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are both dense, with $(\mathbb{R}, <)$ satisfying an additional completeness property that $(\mathbb{Q}, <)$ does not.

Thus $(\mathbb{N}, <)$ is discrete and has a least element, $(\mathbb{Z}, <)$ is discrete with neither least nor greatest element, while $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are both dense, with $(\mathbb{R}, <)$ carrying strictly more order-theoretic structure than $(\mathbb{Q}, <)$.

7.2 The ultrapower of ω

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and consider the ultrapower $\prod_{\mathcal{U}} \omega$. By Łoś's Theorem, $\prod_{\mathcal{U}} \omega$ satisfies all first-order properties of ω. Therefore $\prod_{\mathcal{U}} \omega$ has a least element 0, a successor operation, and is not dense. At the same time, because $\mathcal{U}$ is non-principal, $\prod_{\mathcal{U}} \omega$ contains nonstandard elements represented by non-constant sequences, so it is strictly larger than ω.

Claim 7.2. *There is a substructure of* $(\mathbb{Q}, <)$ *of order type* $\omega^* + \omega$ *inside* $\prod_{\mathcal{U}} \omega$.

Proof. Consider the generalized function sequence (f^t) where $t \in \mathbb{Q}$ and $f^t(x) = \lfloor tx \rfloor$. About each element in f^t we can generate a $(\omega * + \omega)$ by taking (f^t_n) where $n \in \mathbb{N}$ and

$$f^t_n(x) = \begin{cases} f^t(x) - n & \text{if } f^t(x) - n \geq 0 \\ 0 & \text{if } f^t(x) - n < 0 \end{cases}$$

$\lfloor tx \rfloor > \lfloor rx \rfloor$ if $tx - 1 \geq rx$ therefore if $t > r$ then $\forall n, m \in \mathbb{N}$ $f^t_n > f^r_m$ as $f^t_n(x) > f^r_m(x)$ when $tx - n - 1 > qx - m \iff tx - qx > n + 1 - m \iff x > (n + 1 - m)/(t - q)$. Therefore $f^t_n \leq f^r_m$ for finite indices so $f^t_n > f^r_m$ $\square$

The intuition behind the construction is illustrated by the figure below. The key point is that if $t > r$, then the growth of tx eventually dominates the growth of rx. Finite shifts of a given growth function then generate the two-sided pattern $\omega^* + \omega$ about the corresponding class.

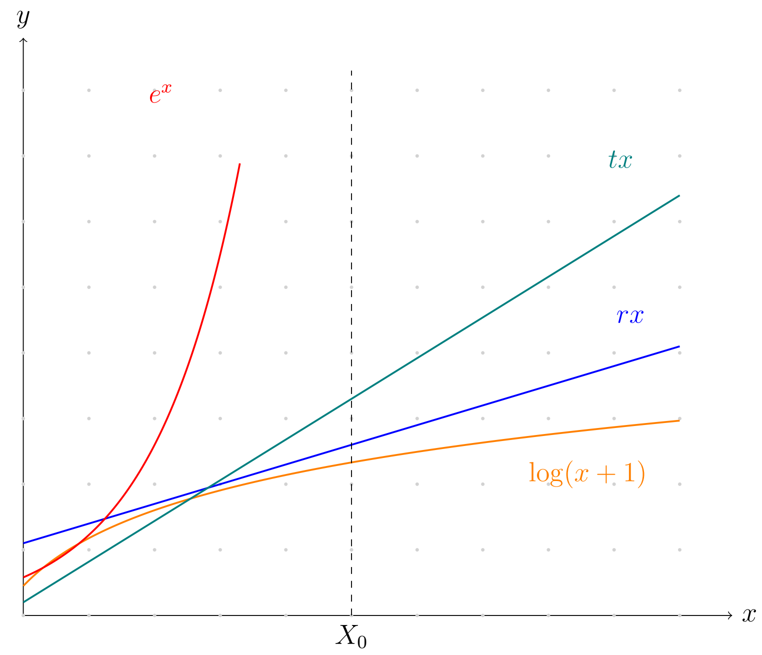

Figure 1: A visualization of eventual domination: larger growth rates eventually dominate smaller ones. In particular, if $t > r$, then tx eventually exceeds rx for all sufficiently large x.

Claim 7.3. *There is a substructure of* $(\mathbb{R}, <)$ *of order type* $\omega^* + \omega$ *inside* $\prod_{\mathcal{U}} \omega$.

Proof. The proof is the same as above, except that we let t range over $\mathbb{R}$ rather than $\mathbb{Q}$. The same eventual domination argument shows that if $t > r$, then for every fixed $n, m \in \mathbb{N}$ we have $f_n^t > f_m^r$. □

As an immediate consequence, $\prod_{\mathcal{U}} \omega$ already contains a large ordered substructure. In particular, since there is such a suborder indexed by $\mathbb{R}$, we obtain $|\prod_{\mathcal{U}} \omega| \geq 2^{\aleph_0}$. On the other hand, $|\prod_{\mathcal{U}} \omega| \leq |\prod \omega| = \aleph_0^{\aleph_0} = 2^{\aleph_0}$. Hence $|\prod_{\mathcal{U}} \omega| = 2^{\aleph_0}$.

Proposition 7.4. *The suborder of* $\prod_{\mathcal{U}} \omega$ *of type* $\omega^* + \omega$ *is dense in the sense that between any two such copies there is another copy of* $\omega^* + \omega$.

Proof. Consider $A = (\omega^* + \omega)_1$ and $B = (\omega^* + \omega)_2$ in $\prod_{\mathcal{U}} \omega$ where for every $f \in A$ and $g \in B$, $f > g$. Let their respective base elements be 0^A and 0^B. All other elements in A and B are $\mathcal{U}$-equivalent to offsets of 0^A_n and 0^B_n, where $0^\alpha_n = \max\{0^\alpha(x) + n, 0\}$ for $n \in \mathbb{Z}$.

Define $0^C = \lfloor (0^A + 0^B)/2 \rfloor$. We claim that 0^C lies strictly between A and B. For arbitrary $n \in \mathbb{Z}$, the elements 0^B_n remain below 0^C, while the elements 0^A_n remain above 0^C, by the same eventual domination argument used above. Thus 0^C determines an element strictly between the two copies.

Now generate a copy of $\omega^* + \omega$ about 0^C by taking the finite offsets 0^C_n. This gives a new copy of $\omega^* + \omega$ lying strictly between A and B. □

The point of this section is that $\prod_{\mathcal{U}} \omega$ is already a nonstandard enlargement of ω: it preserves the first-order theory of $(\mathbb{N}, <)$, but it contains new infinite elements and richer order-theoretic behavior. This is the prototype for the construction used in nonstandard analysis. In the next section, we apply the same ultrapower construction to $\mathbb{R}$ in order to build the hyperreals ${}^*\mathbb{R}$.

8 Nonstandard Analysis

Now, that we have developed some intuition about the size and substructure of the hypernaturals, ${}^*\mathbb{N}$, we move onto construct the hyperreals ${}^*\mathbb{R}$, with ultrapowers. This leads us into the beginning of the development of nonstandard analysis tools. Łoś theorem, as specialized to ${}^*\mathbb{R}$, is often called the transfer principle, which has historical roots with Leibniz's law of continuity.

8.1 Building the Hyperreals, ${}^*\mathbb{R}$

We construct the hyperreals by taking an ultrapower of $\mathbb{R}$, using a fixed nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Define the ultrapower

$$ {}^*\mathbb{R} := \prod_{\mathcal{U}} \mathbb{R}. $$

Write $[x_n]_{\mathcal{U}}$ for the $\mathcal{U}$-equivalence class of a real sequence (x_n). By proposition 6.3 $\mathbb{R}$ embeds into ${}^*\mathbb{R}$ via the diagonal map. As $\mathbb{N} \subset \mathbb{R}$ we also have that ${}^*\mathbb{N} \subset {}^*\mathbb{R}$ and that $\mathbb{N}$ embeds into ${}^*\mathbb{N}$.

8.2 Infinitesimals, the Relation $\approx$, Monads, Standard Parts

Definition 8.1 (infinitesimal)**.** A hyperreal $x \in {}^*\mathbb{R}$ is *infinitesimal* if for every $\varepsilon > 0 \in \mathbb{R}$ we have $|x| < \varepsilon$.

The set of infinitesimals is denoted M_1 and is a ring in ${}^*\mathbb{R}$ [7].
We write $x \approx y$ if $x - y$ is infinitesimal.

Definition 8.2 (Finite)**.** A hyperreal x is *finite* if $|x| < N$ for some real $N > 0$, and *infinite* otherwise.

The set of finite numbers is denoted M_0 and is also a ring [7].

Definition 8.3 (Monad)**.** For $x \in {}^*\mathbb{R}$, the *monad* of x is

$$\mu(x) := \{y \in {}^*\mathbb{R} : y \approx x\}$$

That is, $\mu(x)$ is the set of all hyperreals infinitesimally close to x.

In particular, $\mu(0)$ is precisely M_1, the set of all infinitesimals. More generally, for any real $r \in \mathbb{R}$, the monad $\mu(r)$ consists of all hyperreals of the form $r + \varepsilon$, where ε is infinitesimal. Thus, the monads partition the finite hyperreals into equivalence classes under $\approx$.

Definition 8.4 (Standard part for finite hyperreals)**.** If $x \in {}^*\mathbb{R}$ is finite, then there exists a unique real number $r \in \mathbb{R}$ such that $x \approx r$. We write $r = \mathrm{st}(x)$.

Proof. Omitted ☐

8.3 Internal and External Sets

Within ${}^*\mathbb{R}$, an important distinction is that between *internal* and *external* sets. Internal sets are those that arise directly from the ultrapower construction. In the ultrapower model, every sequence of subsets $(A_i)_{i \in I}$ determines an internal subset $\prod_{\mathcal{U}} A_i$ of the ultraproduct. More generally, internal sets are precisely those objects obtained from standard sets through repeated applications of the nonstandard extension and first-order constructions.

Internal sets play a central role because they are exactly the sets to which the Transfer Principle applies. As a result, internal subsets of $\mathbb{R}$ inherit many of the structural properties enjoyed by ordinary subsets of $\mathbb{R}$. For example, every nonempty internal subset of $\mathbb{N}$ has a least element, by transfer of the well-ordering principle for $\mathbb{N}$.

In contrast, an *external* set is a subset of the nonstandard universe that does not arise from the ultrapower construction. External sets need not satisfy transferred properties and often exhibit behavior impossible for internal sets. Many important objects in nonstandard analysis are external, including the set of infinitesimals, the set of finite hyperreal numbers, and the monad of a standard real number. The externality of these sets is what allows nonstandard analysis to capture notions such as infinitesimal closeness and infinite magnitude that have no direct counterpart in the standard universe.

Proposition 8.5 (Externality of monads)**.** *Let $x \in \mathbb{R}$. The monad of x is an external subset of ${}^*\mathbb{R}$.*

Proposition 8.6 (Externality of the standard reals)**.** *The set $\mathbb{R} \subset {}^*\mathbb{R}$ of standard real numbers is an external subset of ${}^*\mathbb{R}$.*

8.4 Sequences and Limits

Theorem 8.7. *Let $\{s_n\}$ be a standard infinite sequence and let s be a standard real number then s is the limit of $\{s_n\}$ within $\mathbb{R}$, $\lim_{n\to\infty} s_n = s$ in the classical sense if and only if $s_n \approx s$ for all infinite subscripts n. See [7]*

Theorem 8.8. *A standard sequence $\{s_n\}$ converges in $\mathbb{R}$ if and only if $s_n \approx s_m$ for all infinite n and m. See [7]*

8.5 Continuity, Uniform Continuity, Differentiation, and Compactness

Theorem 8.9 (Robinson's continuity criterion [7, Theorem 3.4.5])**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be standard and let $c \in \mathbb{R}$ be standard. Then f is continuous at c if and only if for every $x \in {}^*\mathbb{R}$ with $x \approx c$, ${}^*f(x) \approx f(c)$*

Proof. ($\Rightarrow$) Assume f is continuous at c. Fix $x \in *\mathbb{R}$ with $x \approx c$. Let $\varepsilon > 0$ be real. By continuity, choose $\delta > 0$ such that $|u-c| < \delta \Rightarrow |f(u)-f(c)| < \varepsilon$ for all real u. Since $x \approx c$, we have $|x - c| < \delta$, hence $|f(x) - f(c)| < \varepsilon$. As ε was arbitrary, $*f(x) \approx f(c)$.

($\Leftarrow$) Suppose f is not continuous at c. Then there exists a real $\varepsilon_0 > 0$ such that for every real $\delta > 0$, there exists $u \in \mathbb{R}$ with $|u-c| < \delta$ and $|f(u)-f(c)| \geq$

ε_0. By transfer, for every positive $\delta \in {}^*\mathbb{R}$, there exists $u \in {}^*\mathbb{R}$ with $|u-c| < \delta$ and $|{}^*f(u) - f(c)| \geq \varepsilon_0$. Choose a positive infinitesimal δ. Then there exists $x \in {}^*\mathbb{R}$ such that $|x-c| < \delta$, so $x \approx c$, but $|{}^*f(x)-f(c)| \geq \varepsilon_0$, so ${}^*f(x) \not\approx f(c)$. This contradicts the hypothesis. Therefore f is continuous at c. □

Theorem 8.10 (Nonstandard uniform continuity criterion [7, Theorem 4.5.3])**.** *Let $f : B \to S$ be a standard mapping from a standard metric space B into a standard metric space S. Then f is uniformly continuous on B if and only if for all $p, q \in {}^*B$,*

$$p \approx q \implies {}^*f(p) \approx {}^*f(q).$$

Equivalently, infinitesimally close points in ${}^*B$ have infinitesimally close images in ${}^*S$.

Theorem 8.11. *$c \in \mathbb{R}$ is the derivative of $f(x)$ at x_0 if and only if*

$$\frac{f(x) - f(x_0)}{x - x_0} \approx c$$

for all $x \neq x_0$ in the monad of x_0.

Theorem 8.12 (Compactness and near-standard points [7, Theorem 4.1.13])**.** *Let T be a topological space, and let $\mu(q)$ denote the monad of a standard point $q \in T$. A point $p \in {}^*T$ is called near-standard if there exists a standard point $q \in T$ such that $p \in \mu(q)$. Then T is compact if and only if every point of ${}^*T$ is near-standard.*

Corollary 8.13 (NSA compact subsets [7, Corollary 4.1.15])**.** *Let B be a subset of a topological space T. Then B is compact if and only if for every point $q \in {}^*B$, there exists a standard point $p \in B$ such that $q \in \mu(p)$, where $\mu(p)$ denotes the monad of p.*

8.6 Translation Dictionary: Standard $\leftrightarrow$ Nonstandard

Standard analysis statement	**NSA reformulation in $*\mathbb{R}$**
$a_n \to L$	For every infinite $H \in *\mathbb{N}$, $*a_H \approx L$.
$\lim_{x\to c} f(x) = L$	If $x \approx c$ and $x \neq c$, then $*f(x) \approx L$.
f continuous at c	If $x \approx c$, then $*f(x) \approx f(c)$.
f uniformly continuous on A	If $x, y \in {}^*A$ and $x \approx y$, then ${}^*f(x) \approx {}^*f(y)$.
T compact	Every point of $*T$ is near-standard; that is, for every $p \in *T$ there exists $q \in T$ such that $p \in \mu(q)$.

8.7 An example of an easily proved Theorem through NSA

Theorem 8.14 (Heine–Cantor Theorem)**.** *Let $B \subseteq \mathbb{R}$ be compact, and let $f : B \to \mathbb{R}$ be continuous. Then f is uniformly continuous on B.*

Proof. Let $x, y \in {}^*B$ with $x \approx y$. Since B is compact, every point of ${}^*B$ is near-standard, so $\mathrm{st}(x)$ and $\mathrm{st}(y)$ exist and belong to B. Because $x \approx y$, we have $\mathrm{st}(x) = \mathrm{st}(y)$. Since f is continuous at the standard point $\mathrm{st}(x)$, the nonstandard continuity criterion gives ${}^*f(x) \approx f(\mathrm{st}(x))$. Likewise, ${}^*f(y) \approx f(\mathrm{st}(y))$. As $\mathrm{st}(x) = \mathrm{st}(y)$, it follows that ${}^*f(x) \approx {}^*f(y)$. Therefore, by the nonstandard criterion for uniform continuity, f is uniformly continuous on B. □

9 Ramsey Theory

Another area where ultraproducts find application is, Ramsey Theory, a subset of combinatorics focused on studying emergent regularity within arbitrary structures of a known size. Recent work on Ramsey Theory with ultraproducts can be found here [2].

9.1 Classical Ramsey Theory Example and Formalism

As a brief primer into Ramsey Theory, we will present a real life seminal example of a Ramsey result. Consider, walking into a dinner party of 5 people. You are the sixth. Given that there are six people, it must be the case that a triple of the group all knows each other or that there is another triple of the group which does not know each other. We leave it up to the reader to verify this statement. In addition, 6 people turns out to be the minimum number of people to ensure that a triple of acquaintances or strangers exists. In this sense, at 6 people, a level of regularity emerges within the subsets of people.

Importantly, this notion of knowing each other or not knowing each other, can be translated into the notion of coloring the edges between nodes in a binary fashion with red or blue. And this coloring need not be binary it can include any number of colors, although we will stick with two color colorings for simplicity for now.

All of these notions can be made formal of a complete graph a coloring, and a Ramsey Number:

Definition 9.1 (Complete graph)**.** Let V be a finite set. The **complete graph** on V, denoted K_V (or K_n when $|V| = n$), is the graph whose vertex set is V and in which every pair of distinct vertices is connected by an edge.

Definition 9.2 (Coloring)**.** An n-coloring on a complete graph K_V is a function

$$f : [V]^2 \to \{1, 2, \dots, n\},$$

where

$$[V]^2 = \{\{u, v\} : u, v \in V,\ u \neq v\}$$

denotes the set of all unordered pairs of distinct vertices in V.

Definition 9.3 (Ramsey number)**.** Let $m, n \in \mathbb{N}$. The **Ramsey number** $R(m, n)$ is the least $N \in \mathbb{N}$ such that for every coloring of the edges of the complete graph K_N with two colors, there exists either a subset of vertices of size m whose induced subgraph is monochromatic in the first color, or a subset of vertices of size n whose induced subgraph is monochromatic in the second color.

The definition and the subsequent theorems can be extended to k-color Ramsey numbers $R(x_1, ..x_k)$, but we use the two-color case for simplicity.

9.2 Finite Ramsey from Infinite Theorem

An application of ultraproducts to Ramsey Theory as we will show, is a fast method to prove the Finite Ramsey Theorem from the infinite variation, which is typically considered easier to prove. The finite Ramsey Theorem can be derived from the infinite version using technically complex compactness arguments, however it can also be proved quite easily using ultraproducts. In additon the infinite version can be proved using ultrafilters which we present below.

Theorem 9.4 (Infinite Ramsey Theorem)**.** *For every coloring of the edges of the complete graph on $\mathbb{N}$ with two colors, there exists an infinite subset $H \subseteq \mathbb{N}$ such that all edges between vertices in H have the same color.*

Ultrafilter proof: Consider some blue/red coloring on $\mathbb{N} \times \mathbb{N}$. Formally that is a function $f : [\mathbb{N}]^2 \to \{0, 1\}$.

Now fix a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Define a function $g : \mathbb{N} \to \{0, 1\}$ as $g(n) = 0$ if for $\mathcal{U}$ almost $x \in \mathbb{N}$ (n, x) is red and $g(n) = 1$ if for $\mathcal{U}$ almost $x \in \mathbb{N}$ (n, x) is blue. Given that $\mathcal{U}$ is non-principal; $g(n)$ will be defined for all $n \in \mathbb{N}$ as each subset of N or its compliment will be in the ultrafilter.

Define a sequence as follows. Let $a_0 = 0$, and for $a_0, ...a_{n-1}$ let a_n be the least a st $f(a_i, a) = g(a_i)$ for all $i \leq n$.

For each i the set of x such that $f(a_i, x) = g(a_i)$ is in $\mathcal{U}$. Moreover since $\mathcal{U}$ is closed under finite intersection. $X = \{x : f(a_i, x) = g(a_i), \forall i < n\} \in \mathcal{U}$. Also as $\mathcal{U}$ is non-principal X is infinite hence we can always find such an a.

Proof. □

Now that we've established the Infinite Ramsey theorem, theorem, 9.4 we can prove the stronger and harder Finite Ramsey Theorem from it. Typically such a proof would require a double induction proof or a compactness argument proof, but this can be avoided and great simplified using ultraproducts.

Theorem 9.5 (Finite Ramsey Theorem)**.** *For all $m, n \in \mathbb{N}$, the Ramsey number $R(m, n)$ exists.*

Proof. Fix $m, n \in \mathbb{N}$. We work in the first-order language $L = \{C\}$, where C is a binary relation symbol. We interpret $C(x, y)$ as saying that the edge

$\{x, y\}$ is colored red, and $\neg C(x, y)$ as saying that the edge $\{x, y\}$ is colored blue. Since Ramsey theory colors unordered pairs, we require C to be symmetric, so we include the sentence

$$\mathrm{Sym} := \forall x \forall y \big(C(x, y) \leftrightarrow C(y, x)\big).$$

For reference, the sentence saying that a two-coloring has no red triangle and no blue triangle, i.e. the bad sentence for $R(3, 3)$, is

$$\begin{aligned}\mathrm{Bad}_{3,3} := \forall x_1 \forall x_2 \forall x_3 \Big(&(x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3) \to \\ &(\neg C(x_1, x_2) \vee \neg C(x_1, x_3) \vee \neg C(x_2, x_3))\Big) \\ \wedge \forall y_1 \forall y_2 \forall y_3 \Big(&(y_1 \neq y_2 \wedge y_1 \neq y_3 \wedge y_2 \neq y_3) \to \\ &(C(y_1, y_2) \vee C(y_1, y_3) \vee C(y_2, y_3))\Big).\end{aligned}$$

The first conjunct says that among any three distinct vertices, at least one edge is not red, so there is no red triangle. The second conjunct says that among any three distinct vertices, at least one edge is red, so there is no blue triangle. In general, for fixed m, n, define the first-order sentence $\mathrm{Bad}_{m,n}$ by

$$\begin{aligned}\mathrm{Bad}_{m,n} := \forall x_1 \cdots \forall x_m &\left(\left(\bigwedge_{1 \leq p < q \leq m} x_p \neq x_q \right) \to \left(\bigvee_{1 \leq p < q \leq m} \neg C(x_p, x_q) \right) \right) \\ \wedge \forall y_1 \cdots \forall y_n &\left(\left(\bigwedge_{1 \leq p < q \leq n} y_p \neq y_q \right) \to \left(\bigvee_{1 \leq p < q \leq n} C(y_p, y_q) \right) \right).\end{aligned}$$

The first conjunct says that every m-element subset contains at least one blue edge, so there is no red homogeneous subset of size m. The second conjunct says that every n-element subset contains at least one red edge, so there is no blue homogeneous subset of size n.

Now suppose, for contradiction, that $R(m, n)$ does not exist. Then for every $i \in \mathbb{N}$, there is a two-coloring of a finite complete graph M_i with $|M_i| \geq i$, no red homogeneous subset of size m, and no blue homogeneous subset of size n. Equivalently, for every $i \in \mathbb{N}$, we may choose a finite L-structure M_i such that

$$M_i \models \mathrm{Sym} \wedge \mathrm{Bad}_{m,n}\,.$$

Fix a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and form the ultraproduct

$$M = \prod_{\mathcal{U}} M_i.$$

Since $M_i \models \mathrm{Sym} \wedge \mathrm{Bad}_{m,n}$ for every i, Łoś's Theorem gives

$$M \models \mathrm{Sym} \wedge \mathrm{Bad}_{m,n} .$$

Also, since the finite structures M_i have unbounded size, the ultraproduct M is infinite.

Thus M is an infinite complete graph whose unordered pairs are colored red or blue, but M has no red homogeneous subset of size m and no blue homogeneous subset of size n. This contradicts the Infinite Ramsey Theorem. Indeed, by the Infinite Ramsey Theorem, every infinite two-coloring of pairs has an infinite homogeneous subset. If this homogeneous subset is red, then it contains a red homogeneous subset of size m; if it is blue, then it contains a blue homogeneous subset of size n. Both possibilities contradict $M \models \mathrm{Bad}_{m,n}$. Therefore our assumption was false, and $R(m, n)$ exists. $\square$

Conclusion

The purpose of this thesis was to provide an introduction to ultraproducts and to illustrate some of their wide ranging applications throughout mathematics. Beginning with filters and ultrafilters, we developed the machinery necessary to construct ultraproducts and ultrapowers, culminating in Łoś's Theorem. This theorem provides the fundamental connection between logic and ultraproducts, showing that first-order properties are preserved between an ultraproduct and almost all of its constituent structures.

From this foundation, we explored several applications. We saw how ultraproducts can be used to prove both the countable and uncountable Compactness Theorems, and how these results naturally lead to elementary extensions and the Löwenheim–Skolem Theorem. We then examined the internal structure of ultrapowers, including questions of cardinality and order type, before using ultrapowers of $\mathbb{R}$ to construct the hyperreals and develop several tools from nonstandard analysis. Finally, we considered an application to Ramsey Theory, where ultraproduct methods provide an elegant route from infinite combinatorial statements to finite ones.

The results presented here are largely classical, but the application of ultraproducts to various mathematical disciplines remains active today. Contemporary research continues to apply ultraproducts to problems in model theory, combinatorics, topology, algebra, and other fields. It is my hope that this thesis has provided both an accessible introduction to the subject and a glimpse of the rich mathematical landscape that ultraproducts continue to illuminate.

References

[1] Paul Bankston. Ultraproducts in topology. *General Topology and its Applications*, 13(2):177–191, 1981.

[2] Natasha Dobrinen, Claude Laflamme, and Norbert Sauer. Big ramsey degrees in ultraproducts of finite structures. *Annals of Pure and Applied Logic*, 175(7):103439, 2024.

[3] Rising Entropy. Ultraproducts and compactness, n.d. Accessed: 2026.

[4] Isaac Goldbring. *Ultrafilters Throughout Mathematics.* American Mathematical Society, Providence, RI, 2022.

[5] H. Jerome Keisler. On cardinalities of ultraproducts. *Bulletin of the American Mathematical Society*, 70:644–647, 1964.

[6] H. Jerome Keisler. The ultraproduct construction. *Studies in Logic and the Foundations of Mathematics*, 63:141–177, 1970.

[7] Abraham Robinson. *Non-standard Analysis*, volume 42 of *Studies in Logic and the Foundations of Mathematics.* North-Holland Publishing Company, 1966.