NOETHER–LEFSCHETZ GENERAL COMPLETE INTERSECTION K3 SURFACES OVER THE RATIONALS

A Thesis

Submitted to the Faculty

in partial fulfillment of the requirements for the

degree of

Bachelor of Arts

in

Mathematics

by

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DARTMOUTH COLLEGE

Hanover, New Hampshire

June 2026

Abstract

We prove that the locus of Noether–Lefschetz general polarized K3 surfaces of degree d defined over $\mathbb{Q}$ is Zariski dense in the moduli space for $d \leq 8$. Previously, this was proved by van Luijk in the quartic case, and it follows from work of Elsenhans and Jahnel in the degree 2 case. Innovations on their methods, and employing Mukai's Hodge isometry, suffices to handle the degree 8 case. New methods allow us to deal with the case of degree 6.

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Chapter 1

Introduction

For each even $d \geq 2$, the moduli space $\mathcal{K}_d$ of primitively polarized K3 surfaces of degree d is a 19 dimensional quasiprojective variety. Away from a countably infinite union of divisors in $\mathcal{K}_d$, a polarized K3 surface of degree d has Picard rank 1 over $\mathbb{C}$. These are the Noether–Lefschetz general K3 surfaces. We consider the following open question, cf. [Huy16, Ch. 17, p. 408-409], [Sos10, Remark 1.3.7], [Bra25, Problems 1.4–1.5].

Question 1. *For a given even $d \geq 2$, does there exist a primitively polarized K3 surface S of degree d defined over $\mathbb{Q}$ such that $\mathrm{Pic}(S_{\mathbb{C}}) \cong \mathbb{Z}$?*

In other words, are there $\mathbb{Q}$-points on the Noether–Lefschetz general locus in $\mathcal{K}_d$? Since this locus is the complement of infinitely many divisors, it is not *a priori* clear whether it contains rational points over any given countable field. However, Ellenberg [Ell04] proved that the Noether–Lefschetz general locus in $\mathcal{K}_d$ does admit $\overline{\mathbb{Q}}$-points for every $d \geq 2$. Earlier, results of Terasoma [Ter85] gave a positive answer to Question 1 for $d = 4, 6, 8$, namely, when the general K3 surface of degree d is a complete intersection. We remark that a conjecture of Shafarevich [Sha96], which predicts that only finitely many isomorphisms classes of lattices arise as geometric Picard lattices of K3 surfaces defined over a fixed number field, would imply that

Question 1 has a positive answer for only finitely many d. Shafarevich proved the conjecture for K3 surfaces over a number field of geometric Picard rank 20. Orr and Skorobogatov [OS18], and Orr [OSZ21], proved Shafarevich's conjecture for K3 surfaces of CM type. However, the conjecture seems wide open for K3 surfaces of geometric Picard rank one.

In a similar vein, since $\mathcal{K}_d$ becomes of general type for $d > 122$ by the work of Gritsenko, Hulek, and Sankaran [GHS13, Theorem 6.1], the Bombieri–Lang conjecture would predict that the set of rational points is not Zariski dense, making rational points on the Noether–Lefschetz general locus increasingly rare as d grows. In contrast, when d is small, $\mathcal{K}_d$ is unirational and we expect many rational points. Our main result completes the picture for complete intersection K3 surfaces.

Theorem 1. *The set of Noether–Lefschetz general K3 surfaces defined over $\mathbb{Q}$ is Zariski dense in $\mathcal{K}_d$ for $d \leq 8$.*

Van Luijk [Lui07b] proved this in degree 4 by a pioneering method of leveraging the Weil and Tate conjectures, together with properties of the specialization homomorphism for the Picard group, to construct specific quartic K3 surfaces over $\mathbb{F}_p$ and $\mathbb{F}_{p'}$ with geometric Picard rank 2 and incompatible Picard lattices, forcing a common lift to $\mathbb{Q}$ to have geometric Picard rank 1. Kloosterman [Klo07] showed that in van Luijk's method, the specific K3 surface modulo the second prime p' could be traded for information coming from the Artin–Tate formula. Elsenhans and Jahnel [EJ08a; EJ08b] adapted van Luijk's strategy for K3 surface of degree 2, and then developed a new general technique [EJ11] for K3 surfaces (applying it in degree 2) that only required working modulo a single prime, see also [HV13, Proposition 5.3]. These authors were able to present the first explicit examples of Noether–Lefschetz general K3 surfaces over $\mathbb{Q}$; the case of quartic K3s had been a long-standing challenge due to Mumford, with van Luijk's work yielding the first known explicit examples, see [Vár17, § 2.4-2.6]. As already utilized in [Lui07b, p. 12], the nature of the explicit constructions, by lifting specific K3 surfaces over a finite field, yields the Zariski density in Theorem 1 for $d = 4$, and a similar argument using the construction of Elsenhans and Jahnel [EJ08a; EJ08b] works for $d = 2$, see Section 5.3. For $d = 8$, an argument involving Mukai's degree 8 to degree 2 isogeny (see Section Section 5.2), together with examples first due to Elsenhans and Jahnel [EJ11, § 8], yields the result. Finally, we must further develop the techniques of van Luijk and Elsenhans–Jahnel for the application to K3 surfaces of degree 6.

A K3 surface of degree 6 is a complete intersection $X = X_{2,3} \subset \mathbb{P}^4$. We specifically consider such X whose reduction X_p over $\mathbb{F}_p$ contains a line, and then projection from the line yields a double cover $X_p \to \mathbb{P}^2_{\mathbb{F}_p}$. Generically, this degree 2 model does not contain a tritangent line (see Remark 5.1.5) so we cannot use the techniques of [EJ11] directly. Instead, we prove a generalization of [HV13, Proposition 5.3] to lines contained in projective K3 surfaces, see Section 5.1. After checking $\rho(X_p) = 2$, we lift to a surface $X \subset \mathbb{P}^4$ over $\mathbb{Q}$ that does not contain any lines by a verification using Gröbner bases. The first explicit examples were presented by the author in [Sch].

In degree 8, we use the fact that a complete intersection $X \subset \mathbb{P}^5$ of three quadrics over $\mathbb{Q}$ is Hodge-isogenous to a degree 2 discriminant K3 surface Y over $\mathbb{Q}$, see Section Section 5.2. When the reduction Y_p over $\mathbb{F}_p$ admits a tritangent line and has geometric Picard rank two, yet Y does not admit a tritangent line over $\overline{\mathbb{Q}}$, then Y (and hence X) will have geometric Picard rank one. Explicit examples in degree 8 were (implicitly) constructed using this technique in [EJ11, § 8] and [McK+17, §5].

Finally, in Section 5.3, we show how explicit constructions of K3 surfaces of geometric Picard rank 1 (such as appearing in work of van Luijk, Elsenhans–Jahnel, and in this work) can be used to prove the Zariski density of the locus of Noether–Lefschetz general K3 surfaces over $\mathbb{Q}$ in the moduli space.

Recent work by de Vries [Vri26] gives the first example of a Noether–Lefschetz general K3 surface over $\mathbb{Q}$ of degree 10, but it remains open whether there exist Noether–Lefschetz general K3 surfaces over $\mathbb{Q}$ of any even degree greater than 10 and in which degrees $d > 8$ is the locus of such surfaces Zariski dense in the moduli space $\mathcal{K}_d$.

Acknowledgments. The authors would like to thank Nick Addington for the idea of projecting from a line in the degree 6 case. We also thank Jean-Louis Colliot-Thélène and Bjorn Poonen for discussions concerning Zariski density of rational points as well as Sarah Frei and Salim Tayou for discussions about the specialization homomorphism. Peter Doyle and his friend Steve also provided insightful conversations. This work was partially supported by an Undergraduate Research Assistantship at Dartmouth (URAD) and a William H. Neukom Institute for Computational Science at Dartmouth College research scholarship.

Chapter 2

K3 Surfaces and Other Prerequisites

In this chapter, we begin by defining K3 surfaces. Next, we state some particular theorems for line bundles on K3 surfaces. After that, we prove that every complete intersection K3 surface is either a quartic surface in $\mathbb{P}^3$, the intersection of a quadric and cubic hypersurface in $\mathbb{P}^4$, or the intersection of three quadric surfaces in $\mathbb{P}^5$.

To consider the projective models of K3 surfaces more systematically, we introduce the concept of a polarized K3 surface of degree $2d$ for all positive integers d, and show that the three complete intersection examples are polarized K3 surfaces of degrees 4, 6, and 8 respectively. We then consider the converse, and classify the ways that polarized K3s in these degrees can fail to be complete intersections.

Finally, we prove that suitably nice polarized K3s of degree 2 are double covers of $\mathbb{P}^2$, classify the ways that degree 2 K3s can fail to be double covers, and consider several subtleties in how we can classify them up to isomorphism depending on whether the base field is algebraically closed.

We start by defining a K3 surface:

Definition 2.0.1 (Surface)**.** A *surface* over a field k is a proper, geometrically integral scheme of dimension 2 over k.

Definition 2.0.2 (K3 Surface)**.** A *K3 Surface* X over a field k is a smooth surface over k such that $\Omega^2_{X/k} \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

Section 2.1

Complete Intersection K3s

Any smooth complete surface is projective, hence K3 surfaces are automatically projective. Thus, it makes sense to investigate in what settings they admit a suitably "nice" projective model, for various definitions of "nice". The best possible type of projective model is a complete intersection, especially if we can say that every complete intersection of a given type is a K3. Fortunately, that turns out to be true. Before proving it, we define complete intersections.

Definition 2.1.1 (Complete Intersection)**.** A codimension d closed subscheme $X \subseteq \mathbb{P}^n$ is a *complete intersection* if its homogenous ideal $I(X)$ can be generated by exactly d polynomials $f_1, ..., f_d$. We say that X is a complete intersection of type $(\deg f_1, ..., \deg f_d)$ in $\mathbb{P}^n$.

Remark 2.1.2. These degrees are uniquely determined as an unordered set. Further, we require that $d_i > 1$ for all i because if $d_i = 1$ then the inclusion factors as $X \subseteq \mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ and does not produce a novel complete intersection type.

Theorem 2.1.3. *Let k be any field. A smooth complete intersection surface $X \subseteq \mathbb{P}^n_k$ is a K3 surface if and only if it is of type (4) in $\mathbb{P}^3$, $(2,3)$ in $\mathbb{P}^4$, or $(2,2,2)$ in $\mathbb{P}^5$.*

Before we give the proof, we recall several key ingredients:

Proposition 2.1.4 (Adjunction). *Let Y be a smooth codimension 1 subvariety of a smooth variety X over k. Then*

$$\omega_Y \cong (\omega_X \otimes \mathcal{O}_X(Y))\,|_Y.$$

Proof. See [Har77, Prop. II.8.20]. □

Proposition 2.1.5. *Let k be a field, and let $X \subseteq \mathbb{P}^n_k$ be complete intersection of dimension $q \geq 1$. Then*

$$h^i(X, \mathcal{O}_X(n)) = 0$$

for all $0 < i < q$ and $n \in \mathbb{Z}$.

Proof. See [Har77, Exercise III.5.5]. □

Proof of Theorem 2.1.3. By Proposition 2.1.5, we have that $h^1(X, \mathcal{O}_X) = 0$ for any complete intersection surface X, so it suffices to determine which smooth complete intersection surfaces satisfy $\Omega^2_{X/k} \cong \mathcal{O}_X$. For a smooth projective surface, $\Omega^2_{X/k} \cong \omega_X$, so it suffices to determine when $\omega_X \cong \mathcal{O}_X$.

Consider a complete intersection of type $(d_1, ..., d_k)$ in $\mathbb{P}^n$, and let $(f_1, ..., f_n)$ be the defining polynomials, where $\deg f_i = d_i$. Let $Y_i = V(f_1, \ldots f_i)$. We claim that

$$\omega_{Y_i} \cong \mathcal{O}_{\mathbb{P}^n}\left(-n - 1 + \textstyle\sum_{j=1}^{i} d_j\right)|_{Y_i},$$

and we show this by induction on i.

First, consider Y_1 and note that $\mathcal{O}_{\mathbb{P}^n}(Y_1) = \mathcal{O}_{\mathbb{P}^n}(d_1)$. Moreover, $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n - 1)$, so by Proposition 2.1.4,

$$\omega_{Y_1} \cong (\mathcal{O}_{\mathbb{P}^n}(-n-1) \otimes \mathcal{O}_{\mathbb{P}^n}(d_1))\,|_{Y_1} \cong (\mathcal{O}_{\mathbb{P}^n}(d_1 - n - 1))\,|_{Y_1}.$$

Next, note that $\mathcal{O}_{Y_{i-i}}(Y_i) = \mathcal{O}_{\mathbb{P}^n}(d_i)|_{Y_{i-1}}$, so again by Proposition 2.1.4 we have that

$$\begin{aligned}\omega_{Y_i} &\cong \left(\omega_{Y_{i-1}} \otimes \mathcal{O}_{Y_{i-1}}(Y_i)\right)|_{Y_i} \\ &\cong \left(\mathcal{O}_{\mathbb{P}^n}\left(-n-1+\textstyle\sum_{j=1}^{i-1} d_j\right)|_{Y_{i-1}} \otimes \mathcal{O}_{\mathbb{P}^n}(d_i)|_{Y_{i-1}}\right)|_{Y_i} \\ &\cong \left(\mathcal{O}_{\mathbb{P}^n}\left(-n-1+\textstyle\sum_{j=1}^{i-1} d_j\right) \otimes \mathcal{O}_{\mathbb{P}^n}(d_i)\right)|_{Y_i} \\ &\cong \mathcal{O}_{\mathbb{P}^n}\left(-n-1+\textstyle\sum_{j=1}^{i} d_j\right)|_{Y_i}\end{aligned}$$

and we have proven our claim by induction. Thus, because $X = Y_k$, we have that

$$\omega_X \cong \mathcal{O}_{\mathbb{P}^n}\left(-n-1+\textstyle\sum_{i=1}^{k} d_i\right)|_X.$$

We claim that $\omega_X \cong \mathcal{O}_X$ (and hence X is a K3 surface) if and only if $-n-1+\sum_{i=1}^k d_i = 0$. The "only if" requires proof, i.e. we need to show that if $\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^n}(m)|_X$, then $m = 0$. Equivalently, we need to see that the restriction map $\mathrm{Pic}(\mathbb{P}^n) \to \mathrm{Pic}(X)$ is injective, which is true by the Grothendieck-Lefschetz theorem on Picard groups, so we are done.

Thus, we can simply find the complete intersection types that satisfy

$$-n-1+\textstyle\sum_{i=1}^{k} d_i = 0.$$

Moreover, because X must be a surface, we have that $k = n-2$. If $n = 3$, then $\sum_{i=1}^1 d_i = 4$, so the only possibility is type (3). If $n = 4$, then $\sum_{i=1}^2 d_i = 5$, so the only possibility is type $(2,3)$. If $n = 5$, then $\sum_{i=1}^3 d_i = 6$, so the only possibility is $(2,2,2)$. If $n \geq 6$, then $\sum_{i=1}^{n-2} d_i \geq (n-2)2 > n+1$, so there are no solutions. $\square$

Section 2.2

Line Bundles and Lattices

In this section, we define and recall key properties of the intersection lattice on surfaces in general as well as special properties on K3 surfaces.

Definition 2.2.1. A *lattice* is a pair $(M, .)$, where M is a finitely generated free abelian group and . is a symmetric nondegenerate $\mathbb{Z}$-valued bilinear form on M. We say that M is *unimodular* if the induced map $M \to M^\vee$ is an isomorphism.

Definition 2.2.2. Let X be a proper variety, L a line bundle on X, and C an irreducible curve on X. The *intersection product* $(L.C)$ is $\deg(L|_C)$.

Definition 2.2.3. Let X be a surface, and L_1, L_2 line bundles on X. Then the *intersection product* of L_1 and L_2 is

$$(L_1.L_2) = \chi(\mathcal{O}_X) - \chi({L_1}^{-1}) - \chi({L_2}^{-1}) + \chi({L_1}^{-1} \otimes {L_2}^{-1}).$$

On a smooth surface, these definitions coincide in a reasonable way.

Lemma 2.2.4. *If X is a smooth surface, L is a line bundle, and C an irreducible curve, then $\mathcal{O}_X(C)$ is a line bundle and $(L.\mathcal{O}_X(C)) = (L.C)$*

We recall the following key facts about the intersection product on surfaces without proof.

Lemma 2.2.5. *For any surface X, the intersection product defines a symmetric bilinear form*

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}.$$

Recall that the Néron-Severi group is defined to be $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$, i.e. isomorphism classes of line bundles modulo those algebraically equivalent to zero. By the Néron-Severi theorem, $\mathrm{NS}(X)$ is a finitely generated $\mathbb{Z}$-module for any smooth proper variety X, so $\mathrm{NS}(X)/\mathrm{NS}_{\mathrm{tors}}(X)$ is a finitely generated free $\mathbb{Z}$-module. Moreover, over an algebraically closed field, by the Hodge index theorem, the intersection product gives it the structure of a lattice. Let ρ be the rank of $\mathrm{NS}(X)$.

Theorem 2.2.6 (Hodge Index Theorem)**.** *For any smooth proper variety X over an algebraically closed field k, the intersection form on $\mathrm{NS}(X)/\mathrm{NS}_{\mathrm{tors}}(X)$ has signature $(1, \rho - 1)$.*

Because of the following fact about K3's, we will use Picard rank and Néron-Severi rank interchangeably and consider the Picard group to be a lattice under the intersection form.

Theorem 2.2.7. *On a K3 surface X,*

$$\mathrm{Pic}(X) \cong \mathrm{NS}(X) \cong \mathrm{NS}(X)/\mathrm{NS}_{\mathrm{tors}}(X) \cong \mathbb{Z}^{\rho}.$$

Section 2.3

Nef Line Bundles on K3 Surfaces

Basepoint free line bundles on a scheme are equivalent to maps to projective space, so to study projective models of K3s it makes sense to better understand basepoint free line bundles on K3s. In this section, we broaden this goal slightly and develop a classification of nef line bundles on a K3 surface following the classical work of Saint-Donat.

2.3.1. Nef Line Bundles on Proper Varieties

Before defining classes of line bundles, we need two flavors of intersection product on proper varieties.

Going forward, given an irreducible curve C and a line bundle L on a complete variety X, let $(C.L)$ denote the degree of L restricted to C.

Definition 2.3.1 (Classes of Line Bundles on Proper Varieties)**.** Let X be a proper variety, and L a line bundle on X.

- L is *nef* if $(L.C) \geq 0$ for all irreducible curves $C \subseteq X$.
- L is *basepoint free* if for each point $x \in X$ there exists at least one global section of L which does not vanish at x, equivalently the evaluation morphism $H^0(X, L) \otimes \mathcal{O}_X \to L$ is surjective. In this case, L induces a morphism $\phi_L : X \to \mathbb{P}(H^0(X, L)^\vee)$, and we can define the following additional properties based on properties of the morphism.[1]
  - L is *hyperelliptic* if ϕ_L is a finite morphism of degree 2 onto its image.
  - L is *birational* if ϕ_L is birational onto its image.
  - L is *very ample* if ϕ_L is a closed embedding.
- L is *ample* if there exists some $n > 0$ such that $L^{\otimes n}$ is very ample.

Further, if X is a surface[2],

- L is *big and nef* if L is nef and $L^2 > 0$.

[1] The usage of "hyperelliptic" and "birational" in this sense is only standard for K3 surfaces, but they can technically be defined more generally as we do here.

[2] This definition can be made more generality but it is not this simple. Bigness could also be defined separately from nefness but it should *not* be defined as $L^2 > 0$.

These definitions naturally fall into two kinds: those that are morphism theoretic (defined by properties of ϕ_L), i.e. very ample, birational, and basepoint free, and those that are numerical (defined by the intersection numbers of L with curves), i.e. big and nef and nef. It is not immediately clear whether ample falls into either of these two classes, but the following criterion shows that it is in fact numerical.

Corollary 2.3.2 (Nakai–Moishezon–Kleinman Criterion for Surfaces)**.** *Let L be a line bundle on a projective surface X. Then L is ample if and only if $(L.C) > 0$ for all curves $C \subseteq X$ and $L^2 > 0$.*

We get the following chains of implications on any projective surface[3] (birational is a much less commonly defined property than the rest of these, but we include it because it will be useful for classifying line bundles on K3 surfaces):

Theorem 2.3.3. *Let X be a projective surface and L a line bundle on X. Then the following implications hold for properties of L:*

$$\begin{tikzcd}
& \textit{birational} \arrow[r, Rightarrow, "2"] & \textit{basepoint free} \arrow[dr, Rightarrow, "3"] & \\
\textit{very ample} \arrow[ur, Rightarrow, "1"] \arrow[dr, Rightarrow, "4"] & & & \textit{nef} \\
& \textit{ample} \arrow[r, Rightarrow, "5"] & \textit{big and nef} \arrow[ur, Rightarrow, "6"] &
\end{tikzcd}$$

Remark 2.3.4. Note that the top chain of implications (1, 2) are of morphism-theoretic properties, and the bottom chain (4, 5, 6) are of numerical properties.

Proof of Theorem 2.3.3. Implications 2, 4 and 6 are definitional, so we prove the remaining 4:

[3] We could actually get it in greater generality, but we would need a better definition of big and nef.

1. Any closed embedding is birational onto its image

3. For any curve $C \subseteq X$, pick a global section $s \in H^0(X, L)$ such that s does not vanish on C. Then s is a nonvanishing global section of $L|_C$, so $(L.C) = \deg(L|_C) \geq 0$.

5. By Corollary 2.3.2, $(L.C) > 0$ and $L^2 > 0$, so $(L.C) \geq 0$ and $L^2 > 0$, hence L is big and nef.

$\square$

2.3.2. Saint-Donat's Classification for K3 Surfaces

Now, we consider which additional relations we get when we restrict X to be a K3 surface. It turns out that the following surprising facts hold. Going forward, let X be a K3 surface over a field k and L a line bundle on X.

On K3 surfaces, we get the following stronger version of the Nakai-Moishezon-Kleinman criterion. We define the positive cone $\mathcal{C}_X \subseteq \mathrm{NS}(X)_{\mathbb{R}}$ to be the connected component of the set of classes L with $(L)^2 > 0$ containing an ample class.

Theorem 2.3.5 (Nakai-Moishezon-Kleinman Criterion for K3s [Huy16, Prop 2.1.4])**.** *Assume k algebraically closed. Then L is ample if and only if L is contained in the positive cone $\mathcal{C}_X$ and $(L.C) > 0$ for every smooth rational curve $\mathbb{P}^1 \cong C \subseteq X$.*

Our classification will fundamentally rely on the following result of Saint-Donat.

Theorem 2.3.6 ([Sai74, §4.1])**.** *Assume $k = \overline{k}$ and* $\mathrm{char}(k) \neq 2$. *If $L^2 > 0$ and* $\mathrm{Bs}(|L|)$ *has no one-dimensional components, then $L = \mathcal{O}_X(C)$ for C an irreducible curve and L is basepoint free. Moreover, ϕ_L induces a morphism $X \to \mathbb{P}^n$ where* $n = \frac{1}{2}L^2 + 1$, $\dim \phi_L(X) = 2$, *and exactly one of the following two cases occurs*

- *L is hyperelliptic and the image of ϕ_L has degree $\frac{1}{2}L^2$.*

- *L is birational and the image of ϕ_L has degree L^2.*

Corollary 2.3.7. *Assume $k = \overline{k}$ and $\mathrm{char}(k) \neq 2$. If a line bundle L on a K3 surface is basepoint free and big and nef, then it is either birational or hyperelliptic (but not both).*

Proof. Assume $k = \overline{k}$ and $\mathrm{char}(k) \neq 2$. If L is basepoint free, then $\mathrm{Bs}(|L|)$ is empty, so it certainly has no one-dimensional components. Moreover, if it is big and nef, then $L^2 > 0$, so we can apply Theorem 2.3.6. □

We now summarize a further classification of the birational case.

Theorem 2.3.8. *Assume $k = \overline{k}$ and $\mathrm{char}(k) \neq 2$,*[4] *and let L be birational line bundle on a K3 surface X. Then ϕ_L contracts an irreducible curve $C \subseteq X$ if and only if $(L.C) = 0$. There are only finitely many such curves, and ϕ_L is an isomorphism away from such curves. In particular, ϕ_L is a closed embedding and L is very ample if and only if there are no such irreducible curves on X.*

Proof. See part (iii) of [Sai74, Theorem 6.1] as well as the discussion of θ_L in [Sai74, §4.2]. □

Corollary 2.3.9. *Assume $k = \overline{k}$, $\mathrm{char}(k) \neq 2$. On a K3 surface, a line bundle L is very ample if and only if it is ample and birational.*

Proof. Suppose that a birational line bundle L was ample. Then $(L.C) > 0$ for all irreducible curves C on X, so by Theorem 2.3.8 there are no contracted curves, meaning that ϕ_L is a closed embedding, so L is very ample.

Conversely, suppose that L is very ample. Then L is ample and birational by Theorem 2.3.3. □

[4] This likely holds in all characteristics but Saint-Donat makes this assumption globally and we do not have another reference for this fact in characteristic 2.

A prototypical example of a basepoint free and not big and nef line bundle L is when $L^2 = 0$, i.e. linear system of L defines morphism to $\mathbb{P}^1$, which implies fibers are elliptic curves and L is an *elliptic pencil*. The converse also holds as shown below.

Proposition 2.3.10. *Assume $k = \overline{k}$ and $\operatorname{char}(k) \neq 2$. Let L be basepoint free. If L is not big and nef, then ϕ_L is an elliptic pencil. In particular L is big and nef if and only if the image of ϕ_L is a surface.*

Proof. See [DM89, §2.1], case X_1 for the argument that ϕ_L is an elliptic pencil. On the other hand, L is big and nef, then by Corollary 2.3.7 the image of ϕ_L is a surface. Conversely, suppose that L was not big and nef. Then by Proposition 2.3.10 the image of ϕ_L is not a surface. □

Finally, we consider some results involving the decomposition of a line bundle into its fixed and mobile part. Because of the following lemma, we can refer to line bundles and divisors interchangeably.

Lemma 2.3.11. *Every smooth proper surface (i.e. a smooth proper geometrically integral scheme of dimension 2 over a field k) X is projective and locally factorial, so* $\operatorname{Cl}(X) \cong \operatorname{Pic}(X)$.

Definition 2.3.12 (Fixed and Mobile Parts)**.** Given a line bundle L on a smooth proper surface X, let $|L|$ denote the complete linear system of L, i.e. the set of all effective divisors linearly equivalent to L. Let $\{D_i\}$ be the finite set of prime Weil divisors with positive coefficients in every $D \in |L|$. We define the *fixed part* of L to be the divisor

$$F = \sum_i a_i D_i,$$

where a_i is the minimum coefficient of D_i over all $D \in |L|$. We define the *mobile part* of L to be the line bundle $M = L(-F)$, so $L = F + M \in \operatorname{Pic}(X)$.

Lemma 2.3.13 (Properties of Fixed and Mobile Parts)**.** *Given a line bundle L on a smooth proper surface X, let $L = F + M$ be its decomposition into fixed and mobile parts. Then we have that*

- *F is rigid, i.e. it is the unique effective divisor in its linear equivalence class.*
- *F is the 1-dimensional part of $\mathrm{Bs}\,|L|$, where $\mathrm{Bs}\,|L|$ is defined to be the maximal closed subscheme contained in each $D \subseteq |L|$.*
- *$\mathrm{Bs}\,|M|$ is zero-dimensional.*

For K3's in particular (asumming $k = \overline{k}$ and $\mathrm{char}(k) \neq 2$), it turns out that the base locus of a line bundle has no zero dimensional components by [Sai74, Corollary 3.2], so we have the following.

Lemma 2.3.14. *Let L be a line bundle on a K3 surface X. Then the mobile part M of L is basepoint free.*

Finally, we use this analysis to argue that every nef degree 2 line bundle is represented by an irreducible curve.

Lemma 2.3.15. *Assume $k = \overline{k}$ and $\mathrm{char}(k) \neq 2, 3$. Then a big*[5] *and nef line bundle L on a K3 surface X with $L^2 = 2$ is given by $L = \mathcal{O}_X(C)$ for some irreducible curve C with $p_a(C) = 2$.*

Proof. By [Sai74, Proposition 2.6(i)], it suffices to show that L is basepoint free. Assume to the contrary that it isn't. Then by [Huy16, Corollary 2.3.15] we have that $L = \mathcal{O}_X(mE + C)$ with E a smooth elliptic curve on X and $C \cong \mathbb{P}^1$, and $m \geq 2$. In this case,

$$L^2 = (mE + C)^2 = m^2E^2 + 2mEC + C^2 = 2m - 2 > 2,$$

[5]This is most likely true without the big hypothesis, but this allows us to use a cleaner reference.

a contradiction. □

We can summarize all of these relations for K3 surfaces in Figure 2.1. We will not formalize this, but in some sense this diagram is "not to scale", because ampleness is an open condition and nefness is its closure (so all of the intermediate conditions should be somewhere on the boundary of ample). For now, *unigonal* can be taken to mean a nef line bundle which is not basepoint free. We will give a more natural lattice-theoretic definition in Section 2.4 and prove that they are equivalent.

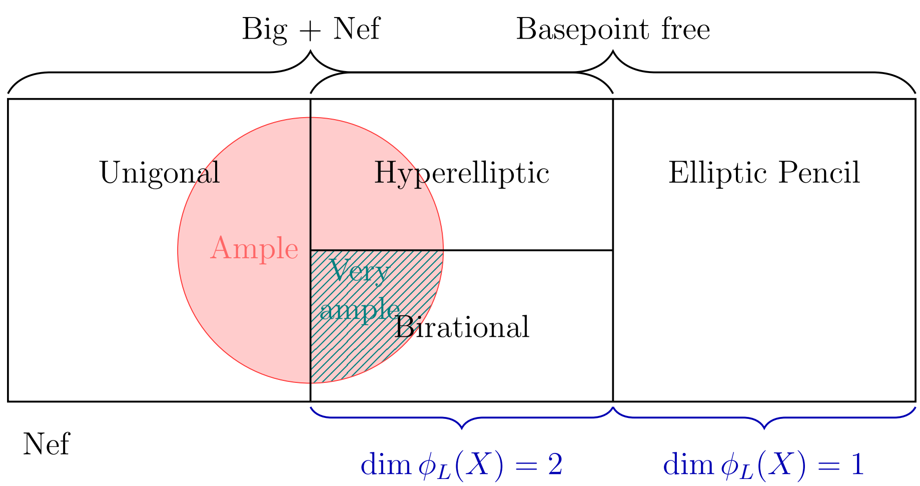

Figure 2.1: Classes of nef line bundles on a K3 surface over k with $k = \overline{k}$ and $\mathrm{char}(k) \neq 2$

We also exctract the following corollary of the above analysis which will be useful for studying polarizations.

Corollary 2.3.16. *Assume $k = \overline{k}$ and $\mathrm{char}(k) \neq 2$. Let L be an ample line bundle on a K3 surface X. Then exactly one of the following holds*

- *L is very ample, equivalently L is birational*
- *L is hyperelliptic*
- *L is unigonal.*

Moreover, we have the following lemma from Riemann-Roch for surfaces and Kodaira vanishing.

Lemma 2.3.17. *Let L be an ample line bundle on a K3 surface X. Then $h^0(X, L) = \frac{1}{2}L^2 + 2$.*

Lemma 2.3.18. *Let C be an irreducible curve on a K3 surface X. Then $p_a(C) = \frac{1}{2}\mathcal{O}_X(C)^2 + 1$.*

Section 2.4

Polarized K3s

In Section 2.1, we gave three examples of K3's embedded in projective space. These can naturally be viewed as a K3 with a choice of a very ample line bundle (given by restricting $O(1)$ to the K3). To study projective models of K3s more systematically, it is useful to weaken this assumption of having a very ample line bundle to having an ample line bundle. This leads us to introduce the concept of a polarization.

Definition 2.4.1 (Polarized K3 Surface)**.** A *polarization* on a K3 surface X is a choice of an ample line bundle $H \in \mathrm{Pic}(X)$ which is a primitive element of the lattice $\mathrm{Pic}(X)$. The *degree* of the polarization is the self-intersection H^2. A *polarized K3 surface of degree d* over k is a pair (X, H) where X is a K3 surface over k and H is a polarization of degree d.

Technically, it is not yet clear that the choices of very ample line bundles implied by Theorem 2.1.3 are polarizations because we have not shown that they are primitive. We now rectify this, and also compute their degrees.

Lemma 2.4.2. *Let k be an algebraically closed field with* $\mathrm{char}(k) \neq 2, 3$. *Given a smooth complete intersection K3 surface X (i.e. of type* (4) *in* $\mathbb{P}^3$*, type* $(2,3)$ *in* $\mathbb{P}^4$*,* *or type $(2,2,2)$ in $\mathbb{P}^5$), restricting $\mathcal{O}(1)$ to X gives a polarization on X of degree 4, 6, or 8, respectively.*

Proof. Consider a complete intersection of type (4) in $\mathbb{P}^3$. Then H^2 is the number of intersection product of two general hyperplane sections of X, i.e. the intersection $(X \cap H_1) \cdot (X \cap H_2) = \#\{X \cap H_1 \cap H_2\} = \#\{X \cap \ell\}$ for some general line $\ell \in \mathbb{P}^3$, so $H^2 = 4$. Suppose that H was not primitive, i.e. there exists some $D \in \mathrm{Pic}(X)$ such that $mD = H$ for some $m > 1$. Then

$$4 = H^2 = (mD)^2 = m^2 D^2,$$

so $m = 2$, hence $D^2 = 1$, but every line bundle on a K3 has even self-intersection.

Similarly, a complete intersection of type $(2,3)$ in $\mathbb{P}^4$ has degree 6, and if $mD = H$, then $6 = m^2 D^2$, which is not possible for integers with $m > 1$.

The same computation shows that the degree of the hyperplane section of a type $(2,2,2)$ complete intersection in P^5 is 8, but the primitivity argument based solely on the self-intersection fails here. In this case it seems possible that $H = 2D$ for some D with $D^2 = 2$. It turns out that if all we know about H is that it is ample, then this is possible. In this case, however, H is very ample, and we can use this additional information to rule this case out. By Lemma 2.3.15, we have that $D = \mathcal{O}_X(C)$ for some irreducible curve C with $p_a(C) = 2$. Thus, by [Sai74, Proposition 5.6], H is hyperelliptic and thus not very ample, a contradiction. $\square$

The next natural question to ask is whether all polarized K3s of degree 4, 6, or 8 are of this form, i.e. whether a primitive ample line bundle on a K3 X is necessarily very ample and induces a morphism to $\mathbb{P}^3$, $\mathbb{P}^4$, or $\mathbb{P}^5$ whose image is a complete intersection of the relevant type. The answer is unfortunately no, but we can cleanly classify the situations in which this can fail, and, because our classification will be purely lattice-theoretic (depending only on properties of $\mathrm{Pic}(X)$ as an abstract lattice), it has a clean moduli space theoretic interpretation.

To clarify what we mean by "lattice-theoretic" conditions, we introduce the following definitions.

Definition 2.4.3. Let M be a lattice and $v \in M$ a primitive element. We say that (M, v) is *k-gonal* if there exists a primitive $w \in M$ such that $(v.w) = k$ and $w^2 = 0$. Specifically, we say that (M, v) is *unigonal*, *hyperelliptic*, or *trigonal*, if it is 1-, 2-, or 3-gonal, respectively.

We say that a polarized K3 surface (X, H) is *unigonal*, *hyperelliptic*, or *trigonal* if the corresponding pair $(\mathrm{Pic}(X), H)$ is.

We have now defined hyperelliptic objects twice: once for line bundles on proper varieties and once for polarized K3 surfaces. These two definitions coincide in a reasonable way, and we check this in the following lemma.

Lemma 2.4.4. *Let (X, H) be a polarized K3 surface with $H^2 \geq 4$ and H basepoint free. Then (X, H) is hyperelliptic (c.f. Definition 2.4.3) if and only if H is hyperelliptic as a line bundle (c.f. Definition 2.3.1).*

Proof. Assume H is a polarization of X, i.e. $H \in \mathrm{Pic}(X)$ is ample and primitive. Suppose H is hyperelliptic as a line bundle. By [Sai74, Theorem 5.2], we have that there exists an irreducible curve E such that $p_a(E) = 1$ and $E.H = 2$ (we can exclude the other case because H is primitive). By Lemma 2.3.18, $\mathcal{O}_X(E)^2 = 0$ for such a curve. We can write $\mathcal{O}_X(E) = mD$ for some primitive class D and $m \geq 1$. Thus

$$2 = \mathcal{O}_X(E).H = (mD).H = m(D.H),$$

so $m = 1$, hence $\mathcal{O}_X(E)$ is primitive and (X, H) is hyperelliptic.

Conversely, suppose that (X, H) is hyperelliptic, i.e. there exists some L such that $L^2 = 0$ and $L.H = 2$. By the analysis of [Sai74, §2.7], we have that L is basepoint-free. Thus, by [Sai74, Proposition 2.6], L is a multiple of $\mathcal{O}_X(E)$ for E an irreducible curve with $p_a(E) = 1$, but L is primitive, so $L = \mathcal{O}_X(E)$, and we are done. □

We also need to check that unigonal polarized K3's match our use in Figure 2.1. The proof is similar to the one above.

Lemma 2.4.5. *Let L be a nef line bundle on a K3 surface X. Then L is not basepoint free if and only if there exists $E \in \mathrm{Pic}(X)$ such that $(E.L) = 1$ and $(E)^2 = 0$. Moreover, a polarization H of X is not basepoint free if and only if (X, H) is unigonal.*

Finally, we give an alternative definition of trigonal.

Lemma 2.4.6. *Let (X, H) be a polarized K3 surface. Then (X, H) is trigonal if and only if there exists an irreducible curve E such that $p_a(E) = 1$ and $(\mathcal{O}_X(E).L) = 3$.*

Next, to be able to compute the projective model given by a line bundle, we will need to take advantage of projective normality.

Definition 2.4.7 (Projective Normality)**.** Let X be a subvariety of $\mathbb{P}^n$ with $L = \mathcal{O}(1)|_X$. We say that X is *projectively normal* if the restriction map

$$H(\mathbb{P}^n, \mathcal{O}(k)) \to H(X, kL)$$

is surjective for all $k \geq 1$.

Lemma 2.4.8. *let X be a projectively normal subvariety of $\mathbb{P}^n$, and I_X be the homogenous ideal of X in $k[x_0, ..., x_n]$, and $L = \mathcal{O}(1)|_X$. Then*

$$\dim(I_X)_m = \binom{n+m}{m} - h^0(X, mL)$$

for all $m \geq 1$*.*

Theorem 2.4.9. *Let* L *be a very ample line bundle on a K3 surface* X*. Then* $\phi_L(X)$ *is projectively normal.*

Proof. Any very ample line bundle is birational and thus basepoint free, so this is a special case of part 2 of [Sai74, Theorem 6.1]. □

2.4.1. Degree 4

Because very amply polarized degree 4 K3s are codimension 1 in $\mathbb{P}^3$, this proof is the most straightforward of the three degrees that can produce complete intersections (degrees 4, 6, and 8). It essentially follows immediately from Saint-Donat's Theorem 2.3.6.

Theorem 2.4.10. *Let* (X, H) *be a polarized K3 surface of degree 4. Then exactly one of the following holds:*

- (X, H) *is hyperelliptic*
- (X, H) *is unigonal*
- H *is very ample and* ϕ_H *is a closed embedding of a quartic hypersurface into* $\mathbb{P}^3$*.*

Proof. By Corollary 2.3.16, Lemma 2.4.4, and Lemma 2.4.5, it suffices to check that if H is very ample, then $\phi_H(X)$ is a quartic hypersurface in $\mathbb{P}^3$.

By Theorem 2.3.6, we have that ϕ_H induces a map to $\mathbb{P}^3$, $\dim \phi_H(X) = 2$, and the degree of $\phi_H(X)$ is $H^2 = 4$. Because $\mathbb{P}^3$ is locally factorial, any codimension 1 subvariety is a hypersurface, so we are done. □

2.4.2. Degree 6

Because degree 6 K3s are not codimension 1 in a projective space, we cannot use local factoriality to deduce that they are a complete intersection. Instead, we have to do a more detailed calculation involving projective normality.

Theorem 2.4.11. *Let (X, H) be a polarized K3 surface of degree 6. Then exactly one of the following holds:*

- *(X, H) is hyperelliptic*
- *(X, H) is unigonal*
- *H is very ample and ϕ_H is a closed embedding of a complete intersection of type $(2, 3)$ into $\mathbb{P}^4$.*

Proof. By Corollary 2.3.16, Lemma 2.4.4, and Lemma 2.4.5, it suffices to check that if H is very ample, then $\phi_H(X)$ is a type (2, 3) complete intersection in $\mathbb{P}^4$.

By Theorem 2.3.6, we have that $X \subseteq \mathbb{P}^4$ because $\frac{1}{2}6 + 2 = 4$. First, we compute the cohomology of H. Because any positive multiple of an ample line bundle is ample, we have by Lemma 2.3.17 that $h^0(X, mH) = 3m^2 + 2$ for all $m \geq 1$.

Let $I \subseteq S = k[x_0, ..., x_n]$ denote the homogenous ideal of X in $\mathbb{P}^n$. By Theorem 2.4.9 X is projectively normal, so by Lemma 2.4.8 we have that

$$\dim I_m = \binom{4+m}{m} - 3m^2 - 2.$$

Thus, $\dim I_1 = 0$, so I has no linear equations, and $\dim I_2 = 1$, so I_2 is generated by a quadric, say q. Next, $\dim I_3 = 6$, but $\dim S_1 I_2 = 5$ because $\dim S_1 = 5$ and multiplication by q is an injective linear map $S_1 \to I_3$. Thus, we have a cubic c in I which is not generated by I_2. Now, we claim that $(q, c) = I$. We could do this via a relatively straightforward geometric argument, but we instead prove it via dimension-counting.

To see this, we need to show that

$$\dim I_m = \dim(I_2 S_{m-2} + I_3 S_{m-3}) = \dim(q S_{m-2} + c S_{m-3})$$

for all $m \geq 4$.

First, note that multiplication by q and c are injective linear maps $S_{m-2} \to S_m$ and $S_{m-3} \to S_m$, respectively. Thus, $\dim qS_{m-2} = \dim S_{m-2} = \binom{m+2}{m-2}$ and $\dim cS_{m-3} = \dim S_{m-3} = \binom{m+1}{m-3}$.

By definition, we have that q does not divide c. Thus, if q and c were not coprime they would have a degree 1 common factor l, but I is prime because all K3 surfaces are irreducible, so $l \in I$, in particular $l \in I_1$, but we showed that $I_1 = 0$, a contradiction. Thus, q and c are coprime so $qS_{m-2} \cap cS_{m-3} = qcS_{m-5}$ if $m \geq 5$ or 0 otherwise, and by a similar analysis as above $\dim qcS_{m-5} = \dim S_{m-5} = \binom{m-1}{m-5}$ when $m \geq 5$.

In the $m = 4$ case, $qS_{m-2} \cap cS_{m-3}$ is 0, so

$$\begin{aligned}
\dim(I_2 S_2 + I_3 S_1) &= \dim qS_2 + \dim cS_1 \\
&= \binom{6}{2} + \binom{5}{1} \\
&= 20 \\
&= 70 - 48 - 2 \\
&= \binom{8}{4} - 3(4^2) - 2 = \dim I_4.
\end{aligned}$$

Now, if $m \geq 5$, we have by inclusion-exclusion that

$$\begin{aligned}
\dim(I_2 S_{m-2} + I_3 S_{m-3}) &= \dim qS_{m-2} + \dim cS_{m-3} - \dim qS_{m-2} \cap cS_{m-3} \\
&= \binom{m+2}{m-2} + \binom{m+1}{m-3} - \binom{m-1}{m-5} \\
&= \binom{m+2}{4} + \binom{m+1}{4} - \binom{m-1}{4} \\
&= \binom{m+4}{4} - 3m^2 - 2 = \dim I_m,
\end{aligned}$$

where the 2nd to last equality follows from expanding all the binomial coefficients as polynomials and checking that the polynomials are identical. □

2.4.3. Degree 8

For degree 6 K3s, every very ample polarization automatically gave us a complete intersection K3, but for degree 8 this is unfortunately no longer the case. Instead, we have to exclude the trigonal locus, leading to the following classification theorem.

Theorem 2.4.12. *Let (X, H) be a polarized K3 surface of degree 8. Then exactly one of the following holds:*

- *(X, H) is hyperelliptic*
- *(X, H) is unigonal*
- *H is very ample and (X, H) is trigonal*
- *H is very ample and ϕ_H is a closed embedding of a complete intersection of type $(2, 2, 2)$ into $\mathbb{P}^5$.*

To prove this, we will rely on the following theorem of Saint-Donat:

Theorem 2.4.13 ([Sai74, Theorem 7.2])**.** *Let L be a birational line bundle on a K3 surface X with $L^2 \geq 8$, and let ϕ_L be the morphism induced by L. Let I be the homogenous ideal of $\phi_L(X) \subseteq \mathbb{P}^n$. Then I is generated in degree 2 except when*

- *There exists an irreducible curve E such that $p_a(E) = 1$ and $(\mathcal{O}_X(E).L) = 3$.*
- *$L \cong \mathcal{O}_X(2B + \Gamma)$ where B is an irreducible curve of genus 2 and γ is an irreducible rational curve such that $(B.\Gamma) = 1$.*

Proof of Theorem 2.4.12. By Corollary 2.3.16, Lemma 2.4.4, and Lemma 2.4.5, it suffices to check that if H is very ample, then $\phi_H(X)$ is a type $(2, 2, 2)$ complete intersection in $\mathbb{P}^4$ if and only if H is not trigonal. Assuming H is very ample, let I be the ideal of X in $\mathbb{P}^5$. If we can show that $\dim I_2 = 3$ and I is generated in degree 2, then we are done.

By Lemma 2.4.6, the first clause in Theorem 2.4.13 is equivalent to (X, H) being trigonal, so it suffices to show that $\dim I_2 = 3$ for very ample H with $H^2 = 8$.

Note that $h^0(X, mL) = 4m^2+2$ for all $m \geq 1$ by Lemma 2.3.17. By Theorem 2.4.9 X is projectively normal, so by Lemma 2.4.8 we have that

$$\dim I_2 = \binom{5+2}{2} - 4(2^2) - 2 = 3,$$

so we are done. $\square$

Section 2.5

Degree 2 Polarized K3s

Degree 2 polarized K3s are not generally complete intersections. Rather, their projective model is as a double cover of $\mathbb{P}^2$ branched over a smooth sextic. There are, however, some subtleties involved in describing them up to isomorphism over an algebraically non-closed field.

Before turning to degree 2 K3s, we review some basic theory of finite morphisms.

Proposition 2.5.1 ([Sta26, Tag 01SA])**.** *Let S be a scheme. Then there is an anti-equivalence of categories*

$$\left\{\begin{matrix} \textit{schemes affine} \\ \textit{over } S \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{quasicoherent sheaves} \\ \textit{of } \mathcal{O}_S\textit{-algebras} \end{matrix}\right\}$$

which associates to $f : X \to S$ the sheaf $f_*\mathcal{O}_X$ and to a quasicoherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ the morphism $\underline{\operatorname{Spec}}_S\mathcal{A} \to S$.

Proposition 2.5.2. *Let S be a scheme and $f : X \to S$ an affine morphism. Then f is finite if and only if $f_*\mathcal{O}_X$ is a finite $\mathcal{O}_S$-module.*

Because all finite morphisms are affine, we then have the following corollary.

Corollary 2.5.3. *Let S be a scheme. Then there is an anti-equivalence of categories*

$$\left\{\begin{matrix} \textit{schemes finite} \\ \textit{over } S \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{finite quasicoherent sheaves} \\ \textit{of } \mathcal{O}_S\textit{-algebras} \end{matrix}\right\}$$

which associates to $f : X \to S$ the sheaf $f_*\mathcal{O}_X$ and to a quasicoherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ the morphism $\underline{\operatorname{Spec}}_S\mathcal{A} \to S$.

We can now use this theory to understand double covers of $\mathbb{P}^2$ using $\mathcal{O}_{\mathbb{P}^2}$-algebras.

Lemma 2.5.4. *Let $f : X \to \mathbb{P}^2$ be a finite degree 2 morphism. Then f is flat and $f_*\mathcal{O}_X$ is a locally free rank-2 $\mathcal{O}_{\mathbb{P}^2}$-algebra.*

Lemma 2.5.5. *Let S be a scheme. Any locally free rank 2 $\mathcal{O}_S$-algebra $\mathcal{A}$ can be given by $\mathcal{A} \cong \mathcal{O}_S \oplus L^{-1}$ for some $L \in \mathrm{Pic}(S)$ with multiplication structure induced by some $s \in H^0(S, L^{\otimes 2})$. Conversely, given a pair (L, s) with $L \in \mathrm{Pic}(S)$ and $s \in H^0(S, L^{\otimes 2})$, the $\mathcal{O}_S$-algebra $\mathcal{O}_S \oplus L^{-1}$ with multiplication structure induced by s is locally free of rank 2.*

Remark 2.5.6. Note that the correspondence of Lemma 2.5.5 is *not* bijective. There are certainly multiple pairs (L, s) which give rise to isomorphic $\mathcal{O}_S$-algebras.

Corollary 2.5.7. *There is are bijections*

$$\left\{\begin{array}{c}\textit{finite flat degree 2}\\ \textit{morphisms to } \mathbb{P}^2\\ \textit{branched over a degree}\\ 2d \textit{ curve}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\mathcal{O}_{\mathbb{P}^2}\textit{-algebras with module}\\ \textit{structure } \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-d)\\ \textit{which are generically reduced}\end{array}\right\} \longleftrightarrow H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2d))^\times / k^{\times 2}$$

where the rightmost set is of nonzero global sections of $\mathcal{O}_{\mathbb{P}^2}(2d)$ up to multiplication by a square in k. Furthermore, a morphism $X \to \mathbb{P}^2$ branched over a curve C of degree $2d$ will correspond to a section $s \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2d))$ such that $V(s) = C$.

Now that we have developed this theory, we can apply it to K3 surfaces.

Proposition 2.5.8. *Given a sextic curve $C \subseteq \mathbb{P}^2$ over an algebraically closed field k, there exists a unique finite 2:1 morphism $X \to \mathbb{P}^2$ over k whose branched locus is C.*

This X is not always a K3 surface, in fact it may be singular, but if we blow up the singularities of X we will get a K3 surface $\tilde{X}$.

Proposition 2.5.9. *Given a sextic curve $C \subseteq \mathbb{P}^2$ over an algebraically closed field k, there exists a unique K3 surface $\tilde{X}$ over k with a morphism $f : \tilde{X} \to \mathbb{P}^2$ which* *factors as a sequence of blowdowns of (-2)-curves followed by a finite flat 2:1 cover of* $\mathbb{P}^2$ *branched over* C*.*

Proposition 2.5.10 (Degree 2 Linear Systems on a K3)**.** *Let* X *K3 surface of degree 2 over a field* k *with* $\operatorname{char}(k) \neq 2$ *and* H *a big and nef and basepoint free divisor. Then* H *determines a dominant morphism* $X \to \mathbb{P}^2$ *which factors as a sequence of blowdowns of* (-2)*-curves followed by a double cover branched over a sextic curve.*

Corollary 2.5.11 (Degree 2 K3 Surfaces)**.** *Let* (X, H) *be a polarized K3 surface of degree 2 over a field* k *with* $\operatorname{char}(k) \neq 2$*. Then* H *determines a dominant morphism* $X \to \mathbb{P}^2$ *which factors as a sequence of blowdowns of* (-2)*-curves followed by a double cover branched over a sextic curve.*

Section 2.6

Specialization

Let X be a K3 surface over $\mathbb{Q}$ and p be a prime. If there exists a proper flat model of X over $\mathbb{Z}$ with a smooth fiber X_p over $\mathbb{F}_p$, we say that p is a prime of *good reduction* for X. We write $\overline{X}_p = X_p \times_{\operatorname{Spec} \mathbb{F}_p} \operatorname{Spec} \overline{\mathbb{F}}_p$. For a prime p of good reduction, there exist *specialization homomorphisms*

$$\operatorname{sp} : \operatorname{NS}(X) \to \operatorname{NS}(X_p) \quad \text{and} \quad \overline{\operatorname{sp}} : \operatorname{NS}(\overline{X}) \to \operatorname{NS}(\overline{X}_p)$$

which are injective and respect the intersection product, cf. [Ful98, §20.3], [Lui07a, Proposition 6.2], [MP12, §3.2]. Moreover, we have the following.

Lemma 2.6.1 (Elsenhans and Jahnel [EJ11, Corollary 3.7])**.** *Let* X *be a proper scheme over* $\mathbb{Q}$ *and* $p > 2$ *be a prime of good reduction. Then the cokernel of the specialization homomorphism* $\overline{\operatorname{sp}} : \operatorname{NS}(\overline{X}) \to \operatorname{NS}(\overline{X}_p)$ *is torsion free.*

Chapter 3

Weil Conjectures and Point Counting

Throughout this chapter, let X be a smooth projective variety over $\mathbb{F}_p$ for some prime p. Our goal will be to describe how the point count of a variety over a finite field is linked the action of the Galois group of $\mathbb{F}_p$ on its étale cohomology groups. In Chapter 4 we will connect this action to a bound on the surface's geometric Picard rank.

Section 3.1

Rational Points

We begin with a moderately pedantic review of rational points of schemes in hopes of rigorously discussing the action of Frobenius (to be defined shortly) on the rational points of a scheme.

Definition 3.1.1 (T-valued points)**.** Let X be an S-scheme. A *T-valued point* of X

is a morphism $T \to X$ of S-schemes, equivalently a commuting triangle

$$\begin{tikzcd} T \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

of scheme morphisms where the vertical arrows are the structure morphisms of T and X. We denote the set of T-valued points of X by $X(T)$.

Definition 3.1.2. Let X be a scheme over k and R a k-algebra. Then a *R-rational point* of X is a $\operatorname{Spec} R$-valued point of X. We denote the set of R-valued points of X by $X(R)$.

Remark 3.1.3. Note that giving R the structure of a k-algebra rather than just a ring specifies a ring homomorphism $k \to R$, which corresponds to a structure morphism $\operatorname{Spec} R \to \operatorname{Spec} k$. This is the $\operatorname{Spec} k$-scheme structure of $\operatorname{Spec} R$ that Definition 3.1.2 is implicitly using.

Lemma 3.1.4. *Let X be a k-scheme and $x \in X$. Then $k(x)$ is a field extension of k and there exists a k-scheme morphism* $\operatorname{Spec} k(x) \to X$.

Proof. We define the k-scheme morphism $(f, f^\#) : \operatorname{Spec} k(x) \to X$. Topologically, let f map the single point of $\operatorname{Spec} k(x)$ to x. On structure sheaves, let $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_{\operatorname{Spec} k(x)}$ be given as follows. For open sets $U \subset X$ with $x \in U$, then $f_*\mathcal{O}_{\operatorname{Spec} k(x)}(U) = k(x)$, so let $f^\#(U)$ be the composition of the natural projections $\mathcal{O}_X(U) \to (\mathcal{O}_X)_x \to k(x)$. For open sets $U \subset X$ with $x \notin U$, $f_*\mathcal{O}_{\operatorname{Spec} k(x)}(U) = 0$, so $f^\#(U)$ is the zero map. This is a sheaf morphism because we can see from how the $(\mathcal{O}_X)_x$ is defined that it commutes with restriction. Thus $(f, f^\#)$ is a morphism of schemes.

To see that $k(x)$ is an extension of k, note that the structure morphism $\pi : X \to \operatorname{Spec} k$ gives a ring homomorphism $\pi^\# : k \to \mathcal{O}_X(X)$, so the composition $f^\#(X) \circ \pi^\# :$

$k \to \mathcal{O}_X(X) \to (\mathcal{O}_X)_x \to k(x)$ gives $k(x)$ is a field homomorphism and must be injective. Call this inclusion i. The induced morphism $i^\# : \operatorname{Spec} k(x) \to \operatorname{Spec} k$ gives $\operatorname{Spec} k(x)$ the structure of a k-scheme.

To see that $(f, f^\#)$ is a morphism of k-schemes (not just a morphism of schemes), it suffices to check that $i^\# = \pi \circ (f, f^\#)$. Because these are morphisms between affine schemes, it is equivalent that the induced ring homomorphisms $k \to k(x)$ are equal, i.e. $i = f^\#(X) \circ \pi^\#$. This was how i was defined, however, so we are done. □

Lemma 3.1.5. *Let L/k be a field extension and X a k-scheme. Then any morphism of k-schemes* $\operatorname{Spec} L \to X$ *factors uniquely through a morphism of k-schemes of the form* $\operatorname{Spec} k(x) \to X$ *for some $x \in X$.*

Proof. Let $i : k \hookrightarrow k(x)$ be the fixed inclusion of k into $k(x)$ given by Lemma 3.1.4, and let $j : k \hookrightarrow L$ be the fixed inclusion of k into L[1], and fix a morphism of k-schemes $f : \operatorname{Spec} L \to X$. The topological image of f is a single point, say $x \in X$. We claim that the morphism f factors through the inclusion $g : \operatorname{Spec} k(x) \hookrightarrow X$ given by Lemma 3.1.4.

Let $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_{\operatorname{Spec} L}$ be the corresponding map of sheaves on X. The map on stalks $(f^\#)_x$ is a local ring homomorphism, so if we let $\mathfrak{m}_x$ be the maximal ideal of $(\mathcal{O}_X)_x$, then $(f^\#)_x(\mathfrak{m}_x) = 0$ because $(f_*(\mathcal{O}_{\operatorname{Spec} L}))_x = L$ is a field. However, $(\mathcal{O}_X)_x/\mathfrak{m}_x \cong k(x)$, so we have an induced morphism $k(x) \to L$. Let $h : \operatorname{Spec} L \to \operatorname{Spec} k(x)$ be the corresponding morphism of affine schemes.

We claim that $f = g \circ h$. This is clear on topological spaces, so it suffices to check it on sheaves, and it in turn suffices to check this on stalks. It is clear from the construction that this holds at $x \in X$, so we have to check the other points of X.

For stalks of points $y \neq x \in X$, we consider two cases. If $y \in \overline{\{x\}}$, then every

[1]We adopt the convention that a field extension includes this data.

open set containing y contains x, so there is a natural map $(\mathcal{O}_X)_y \to (\mathcal{O}_X)_x$ and $(f^\#)_y$ factors as

$$(\mathcal{O}_X)_y \to (\mathcal{O}_X)_x \xrightarrow{(f^\#)_x} (f_*(\mathcal{O}_{\operatorname{Spec} L}))_x \cong (f_*(\mathcal{O}_{\operatorname{Spec} L}))_y.$$

Similarly, $(g^\#)_y$ also factors as

$$(\mathcal{O}_X)_y \to (\mathcal{O}_X)_x \xrightarrow{(g^\#)_x} (f_*(\mathcal{O}_{\operatorname{Spec} k(x)}))_x \cong (f_*(\mathcal{O}_{\operatorname{Spec} k(x)}))_y,$$

so we have that $(f^\#)_y = (g^\#)_y \circ (g_* h^\#)_y$ because $(f^\#)_x = (g^\#)_x \circ (g_* h^\#)_x$

If $y \notin \overline{\{x\}}$, then $(f_*(\mathcal{O}_{\operatorname{Spec} L}))_y = (g_*(\mathcal{O}_{\operatorname{Spec} k(x)}))_y = 0$, so $(f^\#)_y = (g^\#)_y \circ (g_* h^\#)_y$.

Thus, we have successfully factored the point $f : \operatorname{Spec} L \to X$ as a sequence of scheme morphisms

$$\operatorname{Spec} L \xrightarrow{h} \operatorname{Spec} k(x) \xrightarrow{g} X.$$

To see that this factorization is unique, consider two morphisms h_1 and h_2 such that $f = g \circ h_1 = g \circ h_2$. These compositions give the equality

$$(g_* h_1^\#) \circ g^\# = (g_* h_2^\#) \circ g^\#$$

of morphisms of sheaves on X. Taking the stalk at x, we have that

$$(g_* h_1^\#)_x \circ (g^\#)_x = (g_* h_2^\#)_x \circ (g^\#)_x.$$

However, $(g^\#)_x : (\mathcal{O}_X)_x \to (g_* \mathcal{O}_{\operatorname{Spec} k(x)})_x = k(x)$ is simply the quotient map $(\mathcal{O}_X)_x \to (\mathcal{O}_X)_x / \mathfrak{m}_x \cong k(x)$, hence surjective. Thus we can cancel it from the equality and get that

$$(g_* h_1^\#)_x = (g_* h_2^\#)_x.$$

These are equivalent to the global sections ring homomorphisms $h_1^{\#}(\operatorname{Spec} k(x))$ and $h_2^{\#}(\operatorname{Spec} k(x))$, respectively, and h_1 and h_2 are morphisms of affine schemes, so showing that their induced ring homomorphisms are equal is equivalent to showing that they are equal. Thus, we have that f factors uniquely through g.

We wanted to factor f in the category of k-schemes, however, so we still need to check that h is a morphism of k-schemes. Because h is a morphism of affine schemes, this is equivalent to $h^{\#}(\operatorname{Spec} k(x)) \circ i = j$. If we let $\pi : X \to \operatorname{Spec} k$ be X's structure morphism, however, we have that $i = g^{\#}(X) \circ \pi^{\#}(\operatorname{Spec} k)$, so

$$h^{\#}(\operatorname{Spec} k(x)) \circ i = h^{\#}(\operatorname{Spec} k(x)) \circ g^{\#}(X) \circ \pi^{\#}(\operatorname{Spec} k) = f^{\#}(X) \circ \pi^{\#}(\operatorname{Spec} k),$$

but that is equal to j because f is a morphism of k-schemes, so we have shown that it is a k-scheme morphism. □

Proposition 3.1.6. *Let L/k be a field extension and X a k-scheme. Then a L-valued point of X is equivalent to a choice of a point $x \in X$ and a k-algebra homomorphism $k(x) \to L$.*

Proof. An L-valued point f is given by a morphism of k-schemes $f : \operatorname{Spec} L \to X$, so by Lemma 3.1.5 we can factor this through the morphism $g : k(x) \hookrightarrow X$, getting a unique morphism $h : \operatorname{Spec} L \to \operatorname{Spec} k(x)$ of k-schemes. This is equivalent to a k-algebra homomorphism $k(x) \to L$.

Conversely, suppose we had a point $x \in X$ and a k-algebra homomorphism $k(x) \to L$. Then we could get a corresponding k-scheme morphism $\operatorname{Spec} L \to \operatorname{Spec} k(x)$ and compose it with the k-scheme morphism $\operatorname{Spec} k(x) \to X$ implied by Lemma 3.1.4 to get an L-valued point.

To check that these two operations are inverses, suppose that we started with a point and a k-algebra homomorphism $k(x) \to L$. Then we have constructed a morphism $\operatorname{Spec} L \to X$ which factors through the induced morphism $\operatorname{Spec} L \to \operatorname{Spec} k(x)$, and by unique factoring from Lemma 3.1.5 the k-algebra homomorphism we obtain from the technique above must agree with the one we started with.

Conversely, suppose we started with an L-valued point. This direction is clear because we are simply factoring the morphism then composing it back together, so we will end with the same morphism we started with. □

Section 3.2

Frobenius

Our goal is to define an endomorphism of a scheme over a finite field such that fixed points of powers of the endomorphism correspond to rational points in an extension of a given degree. We begin by reviewing Frobenius on finite fields.

Definition 3.2.1 (Frobenius on Finite Fields)**.** Let $\mathbb{F}_q$ be a finite field with q elements, and K an extension of $\mathbb{F}_q$. The *Frobenius element* $\sigma \in \operatorname{Gal}(K/\mathbb{F}_q)$ is the $\mathbb{F}_q$-automorphism of K given by $x \mapsto x^q$.

Proposition 3.2.2 (Galois Theory of Finite Fields)**.**

a) The Galois group $\operatorname{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q)$ *is isomorphic to* $\mathbb{Z}/r\mathbb{Z}$ *and is generated by* σ*.*

b) The field $\overline{\mathbb{F}}_q$ *contains exactly one subfield of order* q^r *for every positive integer* r*, with elements given by*

$$\mathbb{F}_{q^r} = \{x \in \overline{\mathbb{F}}_q \;:\; \sigma^r(x) = x\}.$$

c) The absolute Galois group of $\mathbb{F}_q$ *is the inverse limit*

$$\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \varprojlim_r \mathrm{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q) \cong \varprojlim \mathbb{Z}/r\mathbb{Z} \cong \widehat{\mathbb{Z}}.$$

d) σ *is a topological generator for the profinite group* $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$*.*

Definition 3.2.3 (Absolute Frobenius)**.** Let X be a scheme over a finite field $\mathbb{F}_q$. The *(absolute) Frobenius* endomorphism $F_X : X \to X$ is defined to be the identity on the underlying topological space of X and act by $f \mapsto f^q$ on the structure sheaf. If $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, then we let $\overline{F}_X : \overline{X} \to \overline{X}$ be $F_X \times 1$.

Remark 3.2.4. The notation $\overline{F}_X$ is somewhat strange because it is an endomorphism of $\overline{X}$, not of X. We do not use $F_{\overline{X}}$, however, because this denotes the absolute Frobenius on $\overline{X}$, which is *not* the same as $F_X \times 1$.

Now, we consider various ways that the fixed points of powers of Frobenius correspond to rational points of the scheme. In the end, we will want to describe $\#X(\mathbb{F}_{q^r})$ as an intersection number so as to apply the Lefschetz Trace Formula in Section 3.4. Recall that if X is a scheme over a field k and L/K a field extension, then the natural map $X(K) \to X(L)$ is an injection because any field extension is an injective homomorphism of rings.

Thus, we can consider $X(\mathbb{F}_{q^r})$ to be a subset of $X(\overline{\mathbb{F}}_q)$. One reason for defining Frobenius was to characterize this subset as follows.

Lemma 3.2.5. *Let* X *be a scheme over* $\mathbb{F}_q$*. Then an* $\overline{\mathbb{F}}_q$*-point* $\mathrm{Spec}\,\overline{\mathbb{F}}_q \to X$ *factors through* $\mathrm{Spec}\,\overline{\mathbb{F}}_q \to \mathrm{Spec}\,\mathbb{F}_{q^r}$ *if and only if it is fixed by* F_X^r*.*

Proof. Fix an $f : \mathrm{Spec}\,\overline{\mathbb{F}}_q \to X$, and let x be the topological image of f. Then F_X^r

maps f to the composition

$$g : \operatorname{Spec} \overline{\mathbb{F}}_q \xrightarrow{f} X \xrightarrow{F_X^r} X.$$

As maps of topological spaces, f and g are identical. On the level of structure sheaves, we have the composition

$$g^\# : \mathcal{O}_X \xrightarrow{u \mapsto u^{qr}} \mathcal{O}_X \xrightarrow{f^\#} f_* \mathcal{O}_{\operatorname{Spec} \overline{\mathbb{F}}_q}$$

of maps of sheaves on X.

If $g^\# = f^\#$, then for each open $U \subseteq X$ with $x \in U$, $f^\#(U)(s) = f^\#(U)(s^{qr})$ for all $s \in \mathcal{O}_X(U)$, but $f^\#(U)$ is a ring homomorphism, so $f^\#(U)(s^{qr}) = \left(f^\#(U)(s)\right)^{qr}$. Thus $f^\#(U)(s) \in \operatorname{Spec} \overline{\mathbb{F}}_q$ is fixed by $\sigma^r \in \operatorname{Gal}(\overline{\mathbb{F}}_q / \mathbb{F}_q)$. This holds for all $s \in \mathcal{O}_X(U)$, so $f^\#(U)$ factors through the inclusion

$$f_* \mathcal{O}_{\operatorname{Spec} \mathbb{F}_{q^r}}(U) \cong \mathbb{F}_{q^r} \to \overline{\mathbb{F}}_q \cong f_* \mathcal{O}_{\operatorname{Spec} \overline{\mathbb{F}}_q}(U),$$

i.e. $f^\#$ factors through the sheaf map $f_* \mathcal{O}_{\operatorname{Spec} \mathbb{F}_{q^r}} \to f_* \mathcal{O}_{\operatorname{Spec} \overline{\mathbb{F}}_q}$ where we our viewing $\mathcal{O}_{\operatorname{Spec} \mathbb{F}_{q^r}}$ as a sheaf on $\operatorname{Spec} \overline{\mathbb{F}}_q$ (by pulling back along the map $\operatorname{Spec} \overline{\mathbb{F}}_q \to \operatorname{Spec} \mathbb{F}_{q^r}$ induced by the inclusion). Thus, we have that f factors through $\operatorname{Spec} \overline{\mathbb{F}}_q \to \operatorname{Spec} \mathbb{F}_{q^r}$. The proof of the converse is similar. □

Corollary 3.2.6. *Let X be a scheme over $\mathbb{F}_q$. Then $X(\mathbb{F}_{q^r})$ is identified with the subset of $X\left(\overline{\mathbb{F}}_q\right)$ that are fixed by F_X^r.*

Proof. Note that we are identifying $X(\mathbb{F}_{q^r})$ with its image under the natural injective map $X(\mathbb{F}_{q^r}) \to X\left(\overline{\mathbb{F}}_q\right)$. Because this map is given by pre-composing with $\operatorname{Spec} \overline{\mathbb{F}}_q \to \operatorname{Spec} \mathbb{F}_{q^r}$, it is clear that a rational point is in the image of this map if and only if it factors through $\operatorname{Spec}\overline{\mathbb{F}}_q \to \operatorname{Spec}\mathbb{F}_{q^r}$. □

Recalling the following general lemma, we can phrase this characterization in terms of closed points of $\overline{X}$.

Lemma 3.2.7. *Let X be a scheme of finite type over a field k and $\phi : X \to X$ an endomorphism of X. Then there is a bijection between $X(\overline{k})$ and the closed points of $\overline{X}$ which is equivariant with respect to the action of ϕ.*

Proposition 3.2.8. *Let X be a scheme over $\mathbb{F}_q$. Then the set of closed points of $\overline{X}$ which are fixed by $\overline{F}_X^r$ are in bijection with $X(\overline{F}_{q^r})$.*

Now that we have described the set of $X(\mathbb{F}_{q^r})$ as a set of closed points of $\overline{X}$, we can combine the following ingredients to express it as an intersection number.

Lemma 3.2.9. *If X is a scheme over $\mathbb{F}_q$, then the graph $\Gamma_{\overline{F}_X^r}$ and the diagonal Δ intersect transversely in $\overline{X} \times \overline{X}$.*

Proposition 3.2.10. *Let X be a smooth projective variety over $\mathbb{F}_q$. Then*

$$\#X(\mathbb{F}_{q^r}) = (\Gamma_{\overline{F}_X^r}, \Delta),$$

where the right hand side is the intersection number of the graph of $\overline{F}_X^r$ and the diagonal in $\overline{X} \times \overline{X}$.

Section 3.3

The Weil Conjectures

Here we define the zeta function and state the Weil Conjectures.

Definition 3.3.1 (Hasse-Weil Zeta Function)**.** Let X be a scheme over $\mathbb{F}_q$, and let $N_m = \#X(\mathbb{F}_{q^m})$ for all $m \geq 1$. Then the *(Hasse-Weil) zeta function* of X is

$$Z(X, t) = \exp\left(\sum_{m\geq 1} \frac{N_m}{m} t^m\right)$$

The following four conjectures are the Weil Conjectures.

Theorem 3.3.2 (Rationality)**.** *$Z(X, t)$ is a rational function of t. More precisely,*

$$Z(X, t) = \frac{P_1(t)P_3(t)\cdots P_{2r-1}(t)}{P_0(t)P_2(t)\cdots P_{2r}(t)},$$

where each $P_i(t)$ is an integral polynomial of t where $P_0(t) = 1-t$ and $P_{2r}(t) = 1-q^r t$

Theorem 3.3.3 (Functional Equation and Poincaré Duality)**.** *The zeta function satisfies*

$$Z(X, q^{-r}t^{-1}) = \pm q^{rE/2}t^E Z(X, t)$$

Theorem 3.3.4 (Riemann Hypothesis)**.** *Each $P_i(t) \in \mathbb{Z}[t]$ factors as $\prod_j (1 - \alpha_{ij}t)$ over $\mathbb{C}$ for some $\alpha_{ij} \in \mathbb{C}$ such that $|\alpha_{ij}| = q^{i/2}$ for all $0 \leq i \leq 2r$ and all j.*

Theorem 3.3.5 (Betti Numbers)**.** *If X is a good reduction mod p of a smooth projective variety Y defined over a number field embedded in the complex numbers, then the degree of P_i is the ith Betti number of the space of complex points of Y.*

Section 3.4

Lefschetz Trace Formula

Weil realized that the key ingredient to proving these conjectures is a suitable cohomology theory. The point of this section will be to show that once you have a cohomology theory producing an equivalent of the Lefschetz Trace Formula, Weil's first conjecture, Rationality (Theorem 3.3.2), is purely formal linear algebra.

For now, we will define the *étale cohomology* of X, $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$, by saying that it is a functor

$$H^i_{\text{ét}}(-, \mathbb{Q}_\ell) : \mathsf{Sch} \to \mathsf{Vect}_{\mathbb{Q}_\ell}$$

from schemes to $\mathbb{Q}_\ell$-vector spaces. While this definition technically works for any scheme, you need more hypothesis to get "nice" properties like finite-dimensionality. For now, the point of defining it is that it makes the following version of the Lefschetz trace formula work. We will give a more detailed definition that explains some internals in Chapter 4.

Theorem 3.4.1 (Grothendieck-Lefschetz Trace Formula)**.** *Let Y be a smooth projective variety over an algebraically closed field. Let $\phi : Y \to Y$ be an endomorphism such that the graph Γ_Y intersects the diagonal Δ transversely in $Y \times Y$. Then*

$$(\Gamma_Y.\Delta) = \sum_{r=0}^{2d} (-1)^r \operatorname{Tr}\left(\phi \mid H^r_{\acute{e}t}(Y, \mathbb{Q}_\ell)\right)$$

Once we assume this theorem (the hard part), we can now prove rationality (Theorem 3.3.2) through finite-dimensional linear algebra and formal power series operations.

Proof. See [Mil80, Theorem VI.12.3]. □

Lemma 3.4.2 (c.f. [Mil80, Lemma V.2.7])**.** *Let V be a finite dimensional k-vector space and $\phi : V \to V$ an endomorphism. Then*

$$\log\left(\det\left(1 - \phi t\right)^{-1}\right) = \sum_{m \geq 1} \frac{\operatorname{Tr}(\phi^m)}{m} t^m \in k[[t]].$$

Proposition 3.4.3. *Let k be a field, and consider a sequence of k-vector spaces $V_1, ..., V_{2d}$ for some $d > 0$ with an endomorphism ϕ_r on each V_r for $r = 1, ..., 2d$. If $(a_m)_{m\geq 1}$ is a sequence such that*

$$a_m = \sum_{r=0}^{2d} (-1)^r \operatorname{Tr}(\phi_r^m),$$

then

$$\exp\left(\sum_{m\geq 1} \frac{a_m}{m} t^m\right) = \frac{P_1(t)P_3(t)\cdots P_{2r-1}(t)}{P_0(t)P_2(t)\cdots P_{2r}(t)} \in k[[t]]$$

where $P_i(t) = \det(1 - t\phi_r)$.

Proof. First, note that each $P_i(t)$ has constant term 1 because $\det(1-0\phi_r) = 1$. Thus, any product or quotient of them has constant term 1, so we can take the logarithm

$$\begin{aligned}\log\left(\frac{P_1(t)P_3(t)\cdots P_{2r-1}(t)}{P_0(t)P_2(t)\cdots P_{2r}(t)}\right) &= \sum_{r=0}^{2d} (-1)^{r+1} \log(P_r(t)) \\ &= \sum_{r=0}^{2d} (-1)^r \log\left(\det(1 - t\phi_r)^{-1}\right).\end{aligned}$$

By Lemma 3.4.2,

$$\begin{aligned}\sum_{r=0}^{2d} (-1)^r \log\left(\det(1 - t\phi_r)^{-1}\right) &= \sum_{r=0}^{2d} (-1)^r \sum_{m\geq 1} \frac{\operatorname{Tr}(\phi_r^m)}{m} t^m \\ &= \sum_{m\geq 1} \sum_{r=0}^{2d} (-1)^r \frac{\operatorname{Tr}(\phi^m)}{m} t^m \\ &= \sum_{m\geq 1} \frac{a_m}{m} t^m.\end{aligned}$$

The claim follows by applying exp. □

Proof of Theorem 3.3.2. Apply Proposition 3.4.3 to the variety $Y = \overline{X}$, the pullbacks

$\overline{F}_X^*$ on the vector spaces $H^r_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ and use Theorem 3.4.1 to see that the point counts are equal to the alternating sums of traces of $\overline{F}_X^*$. □

Section 3.5

Counting Points on K3 Surfaces

In this section, we make several remarks and introduce some terminology specific to surfaces or K3 surfaces.

First, let X be a K3 surface over $\mathbb{F}_q$. We have that the dimension of $H^i(\overline{X}, \mathbb{Q}_\ell)$ is 1, 0, 22, 0, 1 for $i = 0, 1, 2, 3, 4$, respectively. Thus, by Theorem 3.3.2 we have that

$$Z(X,t) = \frac{P_1(t)P_3(t)}{P_0(t)P_2(t)P_4(t)},$$

but we always know that $P_0(t) = 1 - t$ and $P_4(t) - 1 - q^r t$, and in this case $\deg P_1 = \deg P_3 = 0$. We know, however, that each $P_i(t)$ is of the form $\det(1 - tF)$ for some linear operator F, so it must have constant term 1 (evaluate it at 0), so $P_1(t) = P_3(t) = 1$. Thus, we have that $Z(X,t)$ (up to constant multple), only depends on $P_2(t)$:

$$Z(X,t) = \frac{1}{(1-t)P_2(t)(1-q^r t)}.$$

Moreover, $\deg P_2(t) = 22$. We call this polynomial the Weil Polynomial of the K3 surface, and it completely determines its Zeta function.

Definition 3.5.1. The *Weil Polynomial* of a smooth projective variety X over $\mathbb{F}_q$ is $\det\left(1 - t\overline{F}_X^*\right)$, were $\overline{F}_X^*$ is the pullback of $\overline{F}_X$ on $H^2_{\text{ét}}(X, \mathbb{Q}_\ell)$.

Going forward, we will treat the Weil Polynomial as a convenient packaging of all the data about the Zeta function and, correspondingly, the point counts of K3 surface. For K3 surfaces in particular, it determines the entire Zeta function (though this is certainly not true in general). It will be our goal in Chapter 4 to prove a bound on the geometric Picard rank of X in terms of properties of the Weil Polynomial.

It turns out that this bound will *not* be specific to K3s, and this makes sense because the Weil Polynomial is defined using H^2, which should correspond to codimension 1 data, i.e. the Picard group, even if it doesn't capture the full data of the Zeta function.

Chapter 4

The Cycle Class Map and Picard Rank Bounds

The goal of this chapter is to show that we can use the Weil Polynomial of a variety X over $\mathbb{F}_p$ to upper bound its geometric Picard rank. This fact fundamentally relies on the interpretation of the Weil Polynomial as the characteristic polynomial of Frobenius on the 2nd étale cohomology group of X. Because we can view isomorphism classes of line bundles $\mathrm{Pic}(X)$ as divisors modulo linear equivalence, these line bundles can also be viewed as codimension one cycles and hence naturally live in $H^2(X)$. The map taking a divisor to its cohomology class is called the "cycle class map".

Section 4.1 The Cycle Class Map for Divisors on Smooth Complex Projective Varieties

First, we review the situation for smooth complex projective varieties because it provides a useful analogy.

Recall cycles up to rational equivalence is equivalent to considering linear equivalence in the case of divisors, which, for *smooth* complex projective varieties, is equivalent to considering isomorphism classes of line bundles because any smooth complex projective variety is locally factorial, so we want a map $\mathrm{Pic}(X) \to H^2_{\mathrm{sing}}(X, \mathbb{Z})$ from the Picard group of X to its singular cohomology.

Recall that given X, we have an associated complex manifold X^{an}, the *analytification* of X, and a morphism of locally ringed spaces $\pi : X^{\mathrm{an}} \to X$. On sheaves on X, we have the *analytification functor* $(-)^{\mathrm{an}} : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_{X^{\mathrm{an}}}}$ given by $\mathcal{F}^{\mathrm{an}} = \pi^* \mathcal{F}$.

Thus, we get the *exponential exact sequence*

$$0 \to \underline{\mathbb{Z}} \xrightarrow{f \mapsto 2\pi i f} \mathcal{O}_{X^{\mathrm{an}}} \xrightarrow{\exp} \mathcal{O}_{X^{\mathrm{an}}}^{\times} \to 1$$

in the category of $\mathcal{O}_{X^{\mathrm{an}}}$-modules.

This gives us a long exact sequence in cohomology, and in particular a connecting map

$$H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}^{\times}) \xrightarrow{c_1} H^2(X^{\mathrm{an}}, \underline{\mathbb{Z}})$$

called the *first Chern class.*

Proposition 4.1.1. *For a complex manifold M, $H^1(M, \mathcal{O}_M^{\times}) \cong \mathrm{Pic}(M)$.*

Proof. We start by computing the Čech cohomology for a fixed open cover $\mathcal{U} = \{U_i\}_{i \in I}$, getting $H^1(\mathcal{U}, \mathcal{O}_M^{\times})$. A 1-cocycle for $\mathcal{U}$ is the data of sections g_{ij} of $\mathcal{O}_M^{\times}$ (i.e. nonvanishing holomorphic functions) on $U_i \cap U_j$ for each g_{ij} which satisfy $g_{ij} g_{jk} g_{ki} = 1$ on all triple intersections. A 1-cocycle thus specifies a holomorphic line bundle L on M with transition functions g_{ij}. A 1-coboundary is a 1-cocycle for which there exist sections h_i for each $i \in I$ such that $g_{ij} = h_j h_i^{-1}$ for all $i, j \in I$. Thus, the h_i glue together to give a nonvanishing global section of L, so line bundles coming from coboundaries are trivial. Finally, we have that the tensor product of line bundles corresponds to the product of their transition functions, so the map from 1-cocycles to line bundles is a group homomorphism which factors through $H^1(\mathcal{U}, \mathcal{O}_M^\times)$, giving an injection $H^1(\mathcal{U}, \mathcal{O}_M^\times) \to \operatorname{Pic}(X)$.

Thus, we have a homomorphism $H^1(M, \mathcal{O}_M^\times) = \varinjlim H^1(\mathcal{U}, \mathcal{O}_M^\times) \to \operatorname{Pic}(X)$ which is still injective. It is also surjective because every holomorphic line bundle is trivialized by some open cover. $\square$

Thus, we have that $H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}^\times) \cong \operatorname{Pic}(X^{\mathrm{an}})$. Next, we will need (a subset of) Serre's GAGA:

Theorem 4.1.2 (GAGA). *For a smooth projective variety X, the analytification functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ restricted to* $\operatorname{Coh}(X)$ *gives an equivalence of categories*

$$\operatorname{Coh}(X) \to \operatorname{Coh}(X^{\mathrm{an}}).$$

Recall that an invertible sheaf on X or X^{an} is coherent. It is also true that the analytification functor preserves tensor products of sheaves, the equivalence of categories gives a group isomorphism $\operatorname{Pic}(X) \to \operatorname{Pic}(X^{\mathrm{an}})$. Thus, we have that $\operatorname{Pic}(X) \cong H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}^\times)$, inducing a map $\operatorname{Pic}(X) \to H^2(X^{\mathrm{an}}, \underline{\mathbb{Z}})$.

Finally, recall that $H^2_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Z}) \cong H^2(X^{\mathrm{an}}, \underline{\mathbb{Z}})$, and $H^2_{\mathrm{sing}}(X, \mathbb{Z})$ is really just an abuse of notation for $H^2_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Z})$, so we have our cycle class map.

Section 4.2

ℓ-adic Cohomology

We will not prove much about ℓ-adic or etale cohomology, and this section serves simply to make sure that the objects are understood with the correct types. Étale cohomology is simply sheaf cohomology in a particular site attached to any scheme, the étale site.

Definition 4.2.1. Given a scheme X, the *small étale site on* X, which we denote by $X_{\text{ét}}$, is the category of étale X-schemes, where a covering of an object $Y \to X$ is given by collections of étale morphisms $\{Y_i \to Y\}_{i \in I}$ such that $\bigsqcup_{i \in I} Y_i \to Y$ is surjective.

Sheaf cohomology was initially developed for sheaves on topological spaces, but it turns out we can define sheaf cohomology as the derived functor cohomology of the global sections functor on essentially any site via the following two facts.

First, we need the following technical hypothesis for derived functor cohomology to work.

Proposition 4.2.2. *Let $\mathcal{C}$ be a site. The category of abelian sheaves on $\mathcal{C}$, denoted* $\operatorname{Sh}\mathcal{C}$*, is an abelian category [Sta26, Lemma 03CN] and has enough injectives [Sta26, Theorem 01DP].*

Second, we need a generalized definition of the global sections functor. We will claim that this exists on arbitrary sites, and give a defintion just for sites with terminal objects (all sites we will use have this property so it suffices for our purposes).

Proposition 4.2.3. *Given any site $\mathcal{C}$, we have a left-exact functor* $\Gamma(-) : \operatorname{Sh}\mathcal{C} \to \mathrm{Ab}$. *If $\mathcal{C}$ has a terminal object X,* $\Gamma(\mathcal{F}) = \mathcal{F}(X)$ *for all* $\mathcal{F} \in \operatorname{Sh}\mathcal{C}$*.*

This matches our traditional definition of global sections of sheaves on a topological space because the space itself is the terminal object in the poset category of open sets under inclusion. In the étale site of a scheme S, the terminal object is the identity étale covering $S \to S$. We can now make the following definition.

Definition 4.2.4. Given a site $\mathcal{C}$ and an abelian sheaf $\mathcal{F} \in \operatorname{Sh}\mathcal{C}$, we define the *$q$-th sheaf cohomology group* $H^q(\mathcal{C}, \mathcal{F})$ to be the qth right derived functor of Γ evaluated at $\mathcal{F}$, $R^q\Gamma(\mathcal{F})$. We define the *q-th Čech cohomology group* $\check{H}^q(\mathcal{C}, \mathcal{F})$ analogously to the definition for sheaves on a topological space.

We can then apply this general definition to the étale site.

Definition 4.2.5. If we have an étale sheaf $\mathcal{F}$ on a scheme X, then the *qth étale cohomology group* of $\mathcal{F}$ is $H^q(X_{\text{ét}}, \mathcal{F})$ and denoted $H^q_{\text{ét}}(X, \mathcal{F})$.

Finally, we can now define ℓ-adic cohomology.

Definition 4.2.6. Given a smooth proper variety X over a separably closed field and some prime ℓ such that $1/\ell \in \mathcal{O}_X$, define the *q-th ℓ-adic cohomology group* of X to be

$$H^q_{\text{ét}}(X, \mathbb{Z}_\ell) = \varprojlim_n H^q_{\text{ét}}(X, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}),$$

where the maps that the limit is being taken over are induced by functoriality of $H^q_{\text{ét}}(X, -)$ and the quotient maps $\mathbb{Z}/\ell^n\mathbb{Z} \to \mathbb{Z}/\ell^m\mathbb{Z}$ as abelian groups.

Remark 4.2.7. Note that it is generally not true that $H^q_{\text{ét}}(X, \mathbb{Z}_\ell) \cong H^q_{\text{ét}}(X, \underline{\mathbb{Z}_\ell})$. The short reason for this is that $H^q_{\text{ét}}(X, -)$ is a left exact functor and does not commute with limits. This is a mild abuse of notation, and we will stick to the following standard throughout: $H^q_{\text{ét}}(X, -)$ will always denote étale cohomology when applied to sheaves. If applied to an abelian group or ring like $\mathbb{Z}_\ell$, it will denote a specifically defined cohomology group like the one above. To make this clear, we will never drop the underline on locally constant sheaves to make clear that the object is a sheaf and not an abelian group or ring.

We now give a basic property of ℓ-adic cohomology and define its twisted variant.

Proposition 4.2.8. *For any smooth proper variety X over a separably closed field k, for any prime $\ell \neq \operatorname{char} k$ and any $i \geq 0$, $H^q_{\text{ét}}(X, \mathbb{Z}_\ell)$ is a finitely generated $\mathbb{Z}_\ell$-module.*

Definition 4.2.9. Let X be a scheme over a field k and let $n \in \mathbb{Z}_{>0}$ such that $1/n \in \mathcal{O}_X$. For $m \in \mathbb{Z}$, define the *Tate twist* $\underline{\mathbb{Z}/n\mathbb{Z}}(m)$ to be a sheaf on $X_{\text{ét}}$ defined as follows:

$$\underline{\mathbb{Z}/n\mathbb{Z}}(m) = \begin{cases} \underline{\mathbb{Z}/n\mathbb{Z}} & \text{if } m = 0, \\ (\mu_n)^{\otimes m} & \text{if } m > 0, \\ \operatorname{Hom}\big((\mu_n)^{\otimes(-m)}, \mathbb{Z}/n\mathbb{Z}\big) & \text{if } m < 0. \end{cases}$$

Let $G = \operatorname{Gal}(k^{\text{sep}}/k)$. Recall that a *Galois module* is a module equipped with an action of a Galois group. We can define the Galois module $\mathbb{Z}/n\mathbb{Z}(m)$ for $m \in \mathbb{Z}$ analogously as above, getting Galois modules $\mathbb{Z}_\ell(m)$ and $\mathbb{Q}_\ell(m)$ by taking an inverse limit and tensoring with $\mathbb{Q}_\ell$.

We define the twisted variant of the ℓ-adic cohomology as follows:

$$H^q_{\text{ét}}(X, \mathbb{Z}_\ell(m)) = \varprojlim_n H^q(X_{\text{ét}}, \underline{\mathbb{Z}/l^n\mathbb{Z}}(m))$$

$$H^q_{\text{ét}}(X, \mathbb{Q}_\ell(m)) = H^q_{\text{ét}}(X, \mathbb{Z}_\ell(m)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

Proposition 4.2.10. *There is an isomorphism of $\mathbb{Q}_\ell$-Galois modules*

$$H^q(X_{k^{\text{sep}}}, \mathbb{Q}_\ell(m)) \cong H^q(X_{k^{\text{sep}}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(m)$$

In particular, this proposition shows that over a separably closed field $H^q(X_{k^{\text{sep}}}, \mathbb{Q}_\ell(m)) \cong H^q(X_{k^{\text{sep}}}, \mathbb{Q}_\ell)$ as $\mathbb{Q}_\ell$-modules (forgetting the Galois action), and in fact $\underline{\mathbb{Z}/n\mathbb{Z}}$ and μ_n are isomorphic as sheaves on $X_{\text{ét}}$.

Section 4.3

ℓ-adic Cycle Class Map for Divisors

As in the complex case, we restrict ourselves to smooth projective varieties. This is not strictly necessary, but we are using the equivalence of line bundles and Weil divisors so we need our schemes to be locally factorial, which smooth projective implies. Before we proceed to define the cycle class map for divisors for ℓ-adic cohomology, we summarize the key steps used in the complex case that we will need to replicate:

1. A ringed space $(X_\bullet, \mathcal{O}_{X_\bullet}^\times)$ such that $H^1(X_\bullet, \underline{A})$ is our desired cohomology for some abelian group A.

2. An exact sequence of sheaves on a site $X_\bullet$ of the form $\underline{A} \to \cdots \to \mathcal{O}_{X_\bullet}^\times$.

3. The isomorphism $H^1(X_\bullet, \mathcal{O}_{X_\bullet}^\times) \cong \mathrm{Pic}(X_\bullet)$

4. A functor $\mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_{X_\bullet}}$ that restricts to an equivalence of categories on invertible sheaves (or some larger subcategory), giving $\mathrm{Pic}(X_\bullet) \cong \mathrm{Pic}(X)$.

As one may expect from the definition of étale cohomology, a ringed space is too much to hope for, but a ringed *site* will do. We start by defining a ringed site and generalizing several concepts to that setting.

Definition 4.3.1. A *ringed site* is a pair $(\mathcal{C}, \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$.

- We say a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ is *quasicoherent* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module which has a global presentation [Sta26, Definition 03DL]. Let $\mathrm{QCoh}(\mathcal{C}, \mathcal{O})$ denote the category of quasicoherent sheaves on $(\mathcal{C}, \mathcal{O})$.

- Let $\mathrm{Pic}(\mathcal{C}, \mathcal{O})$ be the group of invertible sheaves of $\mathcal{O}$-modules up to isomorphism under the tensor product [Sta26, Definition 0409].

It turns out that step (3) is completely general to this setting, and we have the following pair of propositions:

Proposition 4.3.2. *For any ringed site* $(\mathcal{C}, \mathcal{O})$*,* $\check{H}^1(\mathcal{C}, \mathcal{O}^\times) \cong \mathrm{Pic}(\mathcal{C}, \mathcal{O})$

Remark 4.3.3. The Stacks projects states this for locally ringed sites only, but this proof seems to work in greater generality.

Proposition 4.3.4 ([Poo17, Prop. 6.4.12])**.** *If* $\mathcal{F}$ *is an abelian sheaf on a site* $\mathcal{C}$ *with a terminal object, then we have*

$$\begin{aligned}
&\check{H}^0(\mathcal{C}, \mathcal{F}) \xrightarrow{\sim} H^0(\mathcal{C}, \mathcal{F}) = \mathcal{F}(U)\\
&\check{H}^1(\mathcal{C}, \mathcal{F}) \xrightarrow{\sim} H^1(\mathcal{C}, \mathcal{F})\\
&\check{H}^2(\mathcal{C}, \mathcal{F}) \hookrightarrow H^2(\mathcal{C}, \mathcal{F}).
\end{aligned}$$

Step (1) will be fulfilled by the étale site $X_{\text{ét}}$ and the ringed site $(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}})$, and A will (almost) be $\mathbb{Z}/\ell^n\mathbb{Z}$.

Our exact sequence of sheaves will be the Kummer sequence

$$1 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{f \mapsto f^{\ell^n}} \mathbb{G}_m \to 1.$$

This sequence is not exact in the Zariski site, one must consider these to be étale sheaves on the étale site, and this property is an importannt technical feature of the étale site.

Thus, we have a long exact sequence in étale cohomology. In particular, we have the following connecting morphism:

$$H^1(X_{\text{ét}}, \mathbb{G}_m) \xrightarrow{\delta} H^2(X_{\text{ét}}, \mu_{\ell^n})$$

Finally, to satisfy (4) and show that $\text{Pic}(X) \cong \text{Pic}(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}})$, recall that we have a functor $\text{Mod}_{\mathcal{O}_X} \to \text{Mod}_{\mathcal{O}_{X_{\text{ét}}}}$ given by $\mathcal{F} \mapsto ((f : U \to X) \mapsto \Gamma(T, f^*\mathcal{F}))$. We denote this functor $\mathcal{F} \mapsto \mathcal{F}_{\text{ét}}$

Proposition 4.3.5 ((generalization of) Hilbert 90)**.** *For any scheme X, the functor $\mathcal{F} \mapsto \mathcal{F}_{ét}$ restricts to an equivalence of categories* $\text{QCoh}(X) \to \text{QCoh}(X_{ét}, \mathcal{O}_{X_{ét}})$

Proof. See [Sta26, Proposition 03DX]. □

With this proposition, we can analogously conclude that $\text{Pic}(X) \cong \text{Pic}(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}})$, so δ induces a morphism $\text{Pic}(X) \to H^2(X_{\text{ét}}, \mu_{\ell^n})$. Repeating this construction over all n and taking the inverse limit, we have a morphism $\text{Pic}(X) \to H^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))$.

Remark 4.3.6. We should see Proposition 4.3.5 as a generalization of Hilbert 90 for the following reason. Interpret it as (a generalization of) $H^1(X_{\text{ét}}, \mathbb{G}_m) \cong H^1(X, \mathbb{G}_m)$, and let $X = \text{Spec}(K)$ for a field k. Then $\text{Pic}(X) = \{1\}$ because X has only one point, so $H^1(X_{\text{ét}}, \mathbb{G}_m) = 0$, which implies that $H^1(G, L^\times) = \{1\}$ for all Galois extensions L of K.

Section 4.4

Kernel of the cycle class map

Definition 4.4.1. Let X be a scheme. Let $\text{Pic}^\tau(X) \subset \text{Pic}(X)$ be the subgroup of line bundles L such that $(L.C) = 0$ for all curves $C \subseteq X$. We call these the line bundles *numerically equivalent to zero.*

Theorem 4.4.2. *Let X be a smooth projective surface over $\mathbb{F}_p$. Then the kernel of the cycle class map*

$$\mathrm{Pic}(\overline{X}) \to H^2_{\acute{e}t}(\overline{X}, \mathbb{Z}_\ell(1))$$

is $\mathrm{Pic}^\tau(X)$.

Proof. See [Tat65, p. 97-98] □

This is especially nice in our case, because for K3 surfaces $\mathrm{Pic}^\tau(X) = 0$, so the cycle class map is injective.

Section 4.5

Bounding the Picard Rank

As discussed in the previous section, for a K3 surface X_p over $\mathbb{F}_p$, the cycle class map $\mathrm{NS}(\overline{X}_p) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^2_{\text{ét}}(\overline{X}_p, \mathbb{Q}_\ell(1))$ is a Frobenius equivariant injection, cf. [MP12], [Lui07a, Proposition 6.2]. By observing that each divisor class over $\overline{\mathbb{F}}_p$ is defined over some finite extension $\mathbb{F}_{p^m}$ of $\mathbb{F}_p$, each divisor of $\overline{X}_p$ is fixed by F^m for some positive integer m. Thus, if we let $F^*(1)$ be the induced map (of $\mathbb{Q}_\ell$-vector spaces) on $H^2_{\text{ét}}(\overline{X}_p, \mathbb{Q}_\ell(1))$, under the cycle class map, each divisor class becomes an eigenvector of $F^*(1)$ with eigenvalue a root of unity. Thus, by the injectivity of the cycle class map, the $\mathbb{Q}_\ell$-dimension of $\mathrm{NS}(\overline{X}_p) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, and hence the $\mathbb{Z}$-rank of $\mathrm{NS}(\overline{X}_p)$, is bounded above by the number of eigenvalues of $F^*(1)$ that are roots of unity, counted with multiplicity. Eigenvalues of $F^*(1)$ differ from eigenvalues of F^* by a factor of p, so the geometric Picard rank $\rho(\overline{X})$ is bounded above by the number of eigenvalues λ of F^* for which λ/p is a root of unity, counted with multiplicity, cf. [Lui07b, Corollary 2.3]. In fact, a consequence of the Tate conjecture for K3 surfaces, proved by Nygaard–Ogus [NO85], Maulik [Mau14], Charles [Cha13; Cha16], Madapusi Pera [Mad15], Kim–Madapusi Pera [KM16; Mad20], and Ito–Ito–Koshikawa [IIK21], is that the upper bound is tight.

Theorem 4.5.1. *For a K3 surface X defined over $\mathbb{F}_q$, the geometric Picard rank $\rho(\overline{X})$ is equal to the number of eigenvalues λ of F^* for which λ/q is a root of unity, counted with multiplicity.*

Chapter 5

Results

Section 5.1

Degree 6 Case

Let (X, H) be polarized K3 surface of degree 6 defined over $\mathbb{Q}$ that is basepoint free and hyperelliptic, i.e. a smooth complete intersection of type $(2, 3)$ in $\mathbb{P}^4$ over $\mathbb{Q}$ by Theorem 2.4.11.

To bound the geometric Picard rank of X, one could follow the strategy of van Luijk [Lui07b], computing the Weil polynomial of a reduction X_p at a prime of good reduction and applying Theorem 4.5.1 in view of the injectivity of the specialization map. To compute the Weil polynomial, one could hope to count points over $\mathbb{F}_{p^n}$ for $n = 1, \ldots, 12$. However, a naive point counting algorithm that enumerates over all points in $\mathbb{P}^4$ would be on the border of computational feasibility. In theory, there exist p-adic algorithms to compute the Weil polynomial of a complete intersection, which employ the "Cayley trick" to transform a complete intersection to a hypersurface and then use [CHK19], but these are not yet publicly implemented. Instead, in the spirit of [AA18, §3], we leverage the geometry of linear projections to design a fast specialized algorithm.

Lemma 5.1.1. *Let $X \subset \mathbb{P}^4$ be a nondegenerate (a, b) complete intersection surface with hyperplane section H. If the smooth locus of X contains a line L, then projection from L restricts to a dominant generically finite morphism $X \to \mathbb{P}^2$ of generic degree $(a-1)(b-1)$ induced by the linear system of $H - L$.*

Proof. Resolving the projection from L yields a morphism $\phi : \mathrm{Bl}_L \mathbb{P}^4 \to \mathbb{P}^2$. As $L \subset X$ is a smooth divisor in the smooth locus, the strict transform coincides with X, giving an embedding $X \subset \mathrm{Bl}_L \mathbb{P}^4$ and we consider the restriction $\phi|_X : X \to \mathbb{P}^2$. By the construction of the projection map in terms of planes through L, the projective morphism $\phi|_X$ corresponds to the the linear system of $H - L$. The fibers of the projection are the residual intersections of X with planes containing L. Writing $X = X_a \cap X_b$ as a complete intersection of hypersurfaces in $\mathbb{P}^4$ of degree a and b, the residual intersections of X_a and X_b with a plane P containing L are plane curves $C_{a-1}, C_{b-1} \subset P$ of degree $a-1$ and $b-1$, which are both positive since X is nondegenerate. Hence the fiber of $\phi|_X : X \to \mathbb{P}^2$ corresponding to P is the intersection $C_{a-1} \cap C_{b-1}$ of these plane curves. Over the generic fiber, this intersection of plane curves must be finite (otherwise they would share a component and hence X would be birational to a curve bundle over $\mathbb{P}^2$, which is impossible since X is a surface), and hence by Bézout's theorem is a finite scheme of degree $(a-1)(b-1)$. □

In the case of a sextic K3 surface, we can say more.

Lemma 5.1.2. *Let $X \subset \mathbb{P}^4$ be a sextic K3 surface containing a line L. The projection $X \to \mathbb{P}^2$ from L is the composition of a blow-down of (-2)-curves and a finite flat double cover branched over a sextic curve.*

Proof. By Lemma 5.1.1, projection $X \to \mathbb{P}^2$ from L is dominant and induced by the linear system of $H - L$. Since $(H - L)^2 = 6 - 2 \cdot 1 - 2 = 2$ and $H - L$ induces a dominant map $X \to \mathbb{P}^2$, then as explained earlier, it must blow-down (-2)-curves C such that $(H-L).C = 0$ followed by a double cover, which must be branched over a sextic curve. □

Remark 5.1.3. We remark that in the moduli space $\mathcal{K}_6$, the locus of polarized K3 surfaces (X, H) of degree 6 "containing a line" i.e., an irreducible curve $L \subset X$ with $H.L = 1$ and $L^2 = -2$, is the Noether–Lefschetz divisor $\mathcal{K}_{6,1}^{-2}$. Then $H - L$ is effective and $(H-L)^2 = 2$. Moreover, $H - L$ is base point free (and the induced map $X \to \mathbb{P}^2$ is a double cover) if and only if (X, H) is neither unigonal nor hyperelliptic, i.e., is not contained in the Noether–Lefschetz divisors $\mathcal{K}_{6,1}^{0}$ nor $\mathcal{K}_{6,2}^{0}$.

As a consequence, a degree 6 K3 surface containing a line admits a degree 2 model, on which we can use highly optimized Weil polynomial calculation algorithms implemented in `Magma` [BCP97] by Elsenhans and Jahnel.

However, to employ this algorithm, we need to be able to computationally verify that projection from the line is a double cover $X \to \mathbb{P}^2$ and we need to compute an explicit model as a hypersurface of weighted degree 6 in $\mathbb{P}(1,1,1,3)$, or, what will be enough for our purposes (see Remark 5.1.11), at least explicitly compute the sextic branch curve in $\mathbb{P}^2$. To this end, up to a linear change of variables, we can fix $L = V(x_0, x_1, x_2)$ where $(x_0 : x_1 : x_2 : y_0 : y_1)$ are homogenous coordinates for $\mathbb{P}^4$. Assuming that $X = V(f_2, f_3) \subset \mathbb{P}^4$ contains L, where $f_2, f_3 \in k[x_0, x_1, x_2, y_1, y_2]$ are homogenous forms of degree 2 and 3, respectively, then f_2, f_3 can be expressed as

$$\begin{aligned} f_2 &= l_0 y_0 + l_1 y_1 + q \\ f_3 &= l_{00} y_0^2 + l_{01} y_0 y_1 + l_{11} y_1^2 + q_0 y_0 + q_1 y_1 + c \end{aligned} \tag{5.1}$$

where l_0, l_1, l_{00}, l_{01}, and l_{11} are homogenous linear forms, q, q_0, q_1 are homogenous quadratic forms, and c is a homogenous cubic form, all in $k[x_0, x_1, x_2]$.

Theorem 5.1.4. *Let k be a field of characteristic $\neq 2$ and let $X \subset \mathbb{P}^4$ be a sextic K3 surface over k containing a line $L \subset \mathbb{P}^4$. In the notation of (5.1), consider the following symmetric matrix*

$$A = \begin{pmatrix} 0 & v^t \\ v & M \end{pmatrix} = \begin{pmatrix} 0 & l_0 & l_1 & q \\ l_0 & 2l_{00} & l_{01} & q_0 \\ l_1 & l_{01} & 2l_{11} & q_1 \\ q & q_0 & q_1 & 2c \end{pmatrix} \tag{5.2}$$

of homogeneous forms on $\mathbb{P}^2$, write $g_6 = \det(A)$ and $g_3 = \det(M)$, and let $D = V(g_6)$ and $C = V(g_3)$ in $\mathbb{P}^2$. If D is smooth, then projection from L determines a finite flat double cover $\pi : X \to \mathbb{P}^2$ with branch locus D that is tangent to the nodal cubic $C \subset \mathbb{P}^2$.

For a generic choice of coefficients, $X \subset \mathbb{P}^4$ is smooth, $D \subset \mathbb{P}^2$ is smooth, $C \subset \mathbb{P}^2$ has a single node away from D, and C intersects D tangentially in 9 distinct points.

Remark 5.1.5. Before we give the proof, we remark that Hodge theory predicts the existence of a 9-tangent rational cubic, i.e., a rational cubic $C \subset \mathbb{P}^2$ tangent to the branch sextic D in 9 points, in the double cover model of X. Indeed, by Lemma 5.1.1, in a polarized K3 surface (X, H) of degree 6 containing a line L, the double plane model (assuming it exists) is determined by the linear system $H - L$, and we obtain lattice polarizations of degree 6 and 2 on X

X	H	L
H	6	1
L	1	-2

X	$H-L$	L
$H-L$	2	3
L	3	-2

In terms of moduli spaces, projection from L determines a birational map of Noether–

Lefschetz divisors $\mathcal{K}_{6,1}^{-2} \dashrightarrow \mathcal{K}_{2,3}^{-2}$. Finally, we see that in the double cover model, the image of L in $\mathbb{P}^2$ is a rational cubic curve that splits into two rational curve components L and $3(H-L)-L = 3H-4L$ meeting in $L.(3H-4L) = 11$ points on X. The 11 points comprise the preimages of the 9 points of intersection of C and D together with the two distinct preimages of the node on C.

To prove Theorem 5.1.4, we'll need the following consequence of projective duality.

Lemma 5.1.6. *Let k be a field of characteristic $\neq 2$ and $L \subset \mathbb{P}^n$ be a hyperplane dual to $v \in k^{n+1}$. For a smooth quadric $Q \subset \mathbb{P}^n$ with Gram matrix M, we have that L is tangent to Q if and only if $v^t \operatorname{adj}(M) v = 0$, where $\operatorname{adj}(M)$ is the adjugate of M*

When $n = 2$ and Q is a union of two distinct lines in $\mathbb{P}^2$, then $v^t \operatorname{adj}(M) v = 0$ and $v^t M v = 0$ if and only if L is one of the two lines.

And we'll also need the following linear algebra identity.

Lemma 5.1.7. *Let M be an $n \times n$ matrix and v a column vector of length n over an integral domain. Then we have*

$$v^t \operatorname{adj}(M) v = -\det \begin{pmatrix} 0 & v^t \\ v & M \end{pmatrix}$$

where $\operatorname{adj}(M)$ is the matrix adjugate of M.

Proof. Recall that $\operatorname{adj}(M)$ is given by $\operatorname{adj}(M)_{ij} = (-1)^{i+j} M_{ji}$, where M_{ij} is the (i,j)-cofactor of M, i.e., the determinant of M with the ith row and jth column removed. Computing the determinant on the right hand side using cofactor expansion along the first row and then the first column, we get

$$
\begin{aligned}
-\det\begin{pmatrix} 0 & v^t \\ v & M \end{pmatrix} &= -\sum_{i=1}^{n}(-1)^i v_i \sum_{j=1}^{n}(-1)^{j+1} v_j M_{ji} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j} v_i M_{ji} v_j = \sum_{i=1}^{n}\sum_{j=1}^{n} v_i \operatorname{adj}(M)_{ij} v_j = v^t \operatorname{adj}(M) v
\end{aligned}
$$

where $v^t = (v_1, \ldots, v_n)$. $\square$

Proof of Theorem 5.1.4. As in the proof of Lemma 5.1.1, resolving the projection from L yields a morphism $\phi : \widetilde{\mathbb{P}}^4 = \operatorname{Bl}_L \mathbb{P}^4 \to \mathbb{P}^2$. Writing $X = X_2 \cap X_3$ for hypersurfaces $X_2, X_3 \subset \mathbb{P}^4$ of degrees 2 and 3, respectively, we consider the strict transforms $\widetilde{X}_2, \widetilde{X}_3 \subset \widetilde{\mathbb{P}}^4$. We have that $\widetilde{\mathbb{P}}^4 \cong \mathbb{P}(\mathcal{E})$ where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, cf. [EH16, Section 9.3.2], and $\phi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ coincides with the projective bundle map. The homogeneous coordinates $(y_0 : y_1)$ correspond, via the identification $\widetilde{\mathbb{P}}^4 \cong \mathbb{P}(\mathcal{E})$, to a basis of global sections of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Let z be a nonzero global section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \phi^*\mathcal{O}_{\mathbb{P}^2}(-1)$. Then z is unique up to scaling, as

$$
\begin{aligned}
h^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \phi^*\mathcal{O}_{\mathbb{P}^2}(-1)) &= h^0(\mathbb{P}^2, \phi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \mathcal{O}_{\mathbb{P}^2}(-1))) \\
&= h^0(\mathbb{P}^2, \mathcal{E}^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) = 1
\end{aligned}
$$

by the projection formula. Thus $(y_0 : y_1 : z)$ forms a relative system of homogeneous coordinates on $\mathbb{P}(\mathcal{E})$ over $\mathbb{P}^2$. Then $\widetilde{X}_2$ and $\widetilde{X}_3$ can be identified with the subschemes of $\mathbb{P}(\mathcal{E})$ defined by the vanishing of global sections

$$
l_0 y_0 + l_1 y_1 + qz \quad \text{and} \quad l_{00} y_0^2 + l_{01} y_0 y_1 + l_{11} y_1^2 + q_0 y_0 z + q_1 y_1 z + cz^2,
$$

of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \phi^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(3) \otimes \phi^*\mathcal{O}_{\mathbb{P}^2}(1)$, respectively.

This gives an alternate way of considering the proof of Lemma 5.1.1. Indeed, restricting $\phi|_{\widetilde{X}_i} : \widetilde{X}_i \to \mathbb{P}^2$ yields a relative plane curve of degree $i-1$ in $\mathbb{P}(\mathcal{E})$ determined by one of the above equations. Because L is a smooth divisor in the smooth locus of X, we know that X coincides with its strict transform in $\widetilde{\mathbb{P}}^4$, and hence that $\widetilde{X}_2 \cap \widetilde{X}_3 \cong X$. Thus the fiber of $\phi|_X : X \to \mathbb{P}^2$ over a point $a \in \mathbb{P}^2$ is the intersection of the fibers over a of $\phi_{\widetilde{X}_2} : \widetilde{X}_2 \to \mathbb{P}^2$ and $\phi_{\widetilde{X}_2} : \widetilde{X}_2 \to \mathbb{P}^2$. When both relative hypersurfaces are flat at $a \in \mathbb{P}^2$, equivalently, the homogeneous forms

$$l_0(a)y_0+l_1(a)y_1+q(a)z \quad \text{and} \quad l_{00}(a)y_0^2+l_{01}(a)y_0y_1+l_{11}(a)y_1^2+q_0(a)y_0z+q_1(a)y_1z+c(a)z^2,$$

in $(y_0 : y_1 : z)$ are not identically zero, then the fiber is the intersection of a line and a conic in the fiber over a of $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$. Even when one relative hypersurface (or both) is not flat, the fiber above a is still nonempty. In particular, the morphism $\phi|_X : X \to \mathbb{P}^2$ is surjective, and is a double cover away from the non-flat fibers of the individual relative hypersurfaces.

We note that $\phi_{\widetilde{X}_2} : \widetilde{X}_2 \to \mathbb{P}^2$ is not flat over points in $V(l_0, l_1, q) \subset \mathbb{P}^2$ and $\phi_{\widetilde{X}_3} : \widetilde{X}_3 \to \mathbb{P}^2$ is not flat over points in $V(l_{00}, l_{01}, l_{11}, q_0, q_1, c) \subset \mathbb{P}^2$. Clearly, for generic coefficients, both nonflat loci are empty.

Over the locus where both $\widetilde{X}_2$ and $\widetilde{X}_3$ are flat, the double cover $\phi|_X : X \to \mathbb{P}^2$ is ramified precisely when the fiber of $\widetilde{X}_2$ (which is a line in the fibral plane) is tangent to the fiber of $\widetilde{X}_3$ (which is a conic). Lemma 5.1.6 thus shows that this occurs over the vanishing locus of $v^t \operatorname{adj}(M)v$, where M is the bottom right 3×3 submatrix in (5.2) and v is the vector with $v^t = (l_0, l_1, q)$. By Lemma 5.1.7, this is the same as the vanishing locus $D \subset \mathbb{P}^2$ of the determinant of the matrix A in (5.2).

Now, we proceed to show that if D is smooth then the map $\phi|_X : X \to \mathbb{P}^2$ is flat, hence a finite flat double cover of $\mathbb{P}^2$ branched over D. By Jacobi's formula, we have that

$$\frac{\partial}{\partial x_i}\det(A) = \operatorname{tr}\left(\operatorname{adj}(A)\frac{\partial A}{\partial x_i}\right)$$

and we'll proceed to use the jacobian criterion to show that D is singular at points over which $\phi|_X : X \to \mathbb{P}^2$ is not flat, equivalently, either one of the $\widetilde{X}_i \to \mathbb{P}^2$ is not flat or the fiber of $\widetilde{X}_2$ (the line) is a component of the fiber of $\widetilde{X}_3$ (which is hence a union of lines). Over a point $a \in \mathbb{P}^2$ where $\widetilde{X}_2$ is not flat, the first rown and column of A is uniformly zero, hence $\operatorname{adj}(A)$ is at most nonzero only in the $(1,1)$-entry. But this multiplied by $\frac{\partial A}{\partial x_i}$ is the zero matrix. Similary, where $\widetilde{X}_3$ is not flat, then the entire lower right 3×3 block of A is zero, meaning that already $\operatorname{adj}(A)$ is the zero matrix. Over a point $a \in \mathbb{P}^2$ (which we can assume is a geometric point since we are testing smoothness) where $\widetilde{X}_3$ is a union of distinct lines, one of which is $\widetilde{X}_2$, we can change variables in the fibral plane with homogeneous coordinate $(x : y : z)$ so that $\widetilde{X}_3 = V(xy)$ and $\widetilde{X}_2 = V(x)$. In these variables, Lemma 5.1.6 says that $v^t = (1,0,0)$ or $v^t = (0,1,0)$, in which case $\operatorname{adj}(A)$ is the zero matrix. Where $\widetilde{X}_3$ is the double line $\widetilde{X}_2$, we can change variables in the fibral plane with homogeneous coordinate $(x : y : z)$ so that $\widetilde{X}_3 = V(x^2)$ and $\widetilde{X}_2 = V(x)$ and $v^t = (1,0,0)$, in which case $\operatorname{adj}(A)$ is again the zero matrix. Hence in all cases, D is singular at a. This shows that the smoothness of D implies the flatness of the double cover, and it has already been noted that the branch locus will then be D.

As for the tangency of the intersection of D and C, this is a standard fact about the resolution of 9 points in $\mathbb{P}^2$ and the shape of the matrix A in (5.2), cf. [Bea00, Section 4.1]. The fact that C has a node can be checked directly.

Finally, in the space of all coefficients, the required smoothness and intersection conditions are open, so it suffices to find a single case where they all occur simultaneously, e.g., the example appearing in Corollary 5.1.10. □

Remark 5.1.8. The fact that the branch locus of the degree 2 model of a sextic K3 surface containing a line has a symmetric determinental presentation of the form $\det A$ where A is the 4×4 graded-homogeneous matrix of homogeneous forms on $\mathbb{P}^2$ in (5.2) is indicative that there should be a relationship with cubic fourfolds containing a plane. Indeed, the matrix A determines a cubic fourfold $Z \subset \mathbb{P}^5$ containing a plane, which (in the notation of (5.1)) has equation $f_3 + y_3 f_2 = 0$ if $(x_0 : x_1 : x_2 : y_1 : y_2 : y_3)$ are homogeneous coordinates on $\mathbb{P}^5$. Hence Z is singular at $(0 : 0 : 0 : 0 : 0 : 1)$ and generically this singularity is an ordinary double point. As in [Has00, Section 4.2], the sextic K3 surface X appears in the resolution of the projection $Z \dashrightarrow \mathbb{P}^4$ from the node as the locus in $\mathbb{P}^4$ of lines through the node. In this case, the tangent cone of Z at the node coincides with the projective cone in $\mathbb{P}^5$ over the quadric $V(f_2) \subset \mathbb{P}^4$. Note that Z also contains the plane $V(x_0, x_1, x_2) \subset \mathbb{P}^5$. Hence the locus of these cubic fourfolds forms a component of $\mathcal{C}_6 \cap \mathcal{C}_8$, where $\mathcal{C}_6, \mathcal{C}_8 \subset \mathcal{C}$ are the Hassett divisors parameterizing cubic fourfolds admitting a node and containing a plane, respectively, in the moduli space $\mathcal{C}$ of (semi-stable) cubic fourfolds, see [Has00]. However, this is a different component in $\mathcal{C}_6 \cap \mathcal{C}_8$ from the one considered in [Ste03], since the sextic branch divisor of the associated K3 surface of degree 2 is actually smooth for the very general element. We record that the intersection form on the lattice of integral type $(2, 2)$ Hodge classes in the limiting Hodge structure associated to the nodal cubic fourfold Z has a lattice polarization by

Z	h^2	P	X
h^2	3	1	6
P	1	3	1
X	6	1	14

where h^2 is the square of the hyperplane class, P is the class of the plane, and X is the class of the sextic K3 surface.

Now, we proceed to use the above to establish sufficient criteria for a sextic K3 surface over $\mathbb{Q}$ to have geometric Picard rank 1. The following general criterion for a projective K3 surface to have geometric Picard rank 1 is inspired by [EJ11, Example 1.7], [HV13, Proposition 5.3], and [McK+17, Proposition 26].

Theorem 5.1.9. *Let $X \subset \mathbb{P}^n_{\mathbb{Q}}$ be a K3 surface. Assume that X contains no lines over $\overline{\mathbb{Q}}$, and that for some prime $p > 2$ of good reduction X_p contains a line and $\rho(\overline{X}_p) = 2$. Then $\rho(\overline{X}) = 1$.*

Proof. Let $H \in \mathrm{NS}(X)$ be the hyperplane section. Assume, to get a contradiction, that X has geometric Picard rank at least 2. If $\rho(\overline{X}) > 2$, we have a contradiction because $\overline{\mathrm{sp}} : \mathrm{NS}(\overline{X}) \to \mathrm{NS}(\overline{X}_p)$ is injective and we are assuming $\rho(\overline{X}_p) = 2$. Thus, we can assume that $\rho(\overline{X}) = 2$. Because $\overline{\mathrm{sp}}$ is injective, the image of $\overline{\mathrm{sp}}$ has rank 2 as well, so the cokernel has rank 0. However, by Lemma 2.6.1, the cokernel is torsion free, so it must be 0. Thus $\overline{\mathrm{sp}}$ is surjective.

Finally, let L be a line on X_p. Because $\overline{\mathrm{sp}}$ is surjective, we have a divisor class $L' \in \mathrm{NS}(\overline{X})$ such that $\overline{\mathrm{sp}}(L') = L$, but $\overline{\mathrm{sp}}$ preserves degree (i.e., intersection with H), so L' is the class of a line on $\overline{X}$, contradicting our hypothesis. □

We remark that for a K3 surface that is a double cover $X \to \mathbb{P}^2$, a "line" on X, i.e., a divisor class $L \in \mathrm{Pic}(X)$ with $H.L = 1$ and $L^2 = -2$, is precisely a component of the preimage of a tritangent line to the branch divisor of the cover. Hence, one can view Theorem 5.1.9 as an amply polarized generalization to any degree of the strategy initiated in [EJ11, Example 1.7], cf. [HV13, Proposition 5.3], for degree 2 K3 surfaces.

To construct an explicit example, we randomly generate sextic K3 surfaces X_p over $\mathbb{F}_p$ containing a fixed line $L \subset \mathbb{P}^4$ until we find one with $\rho(\overline{X}_p) = 2$, which we verify by calculating the Weil polynomial using projection from the line. Then, we lift X_p to X over $\mathbb{Q}$ in such a way that $\overline{X}$ contains no lines, and then apply Theorem 5.1.9. We implemented this strategy to find the following.

Corollary 5.1.10. *Let $X = V(f_2, f_3) \subset \mathbb{P}^4_{\mathbb{Q}}$ where*

$$\begin{aligned} f_2 &= x_0^2 - 3x_0x_1 + 3x_1^2 + 5x_0x_2 + 4x_1x_2 + 5x_2^2 - x_0x_3 - 2x_1x_3 \\ &\quad - 3x_2x_3 - 5x_0x_4 + 5x_1x_4 + 47x_3^2 + 47x_4^2 \\ f_3 &= 2x_0^3 + 3x_0^2x_1 + 3x_0x_1^2 + x_1^3 - x_0x_1x_2 - 3x_1^2x_2 + 4x_0x_2^2 - 4x_1x_2^2 + 5x_2^3 \\ &\quad + 4x_0^2x_3 + x_0x_1x_3 + 5x_1^2x_3 + 4x_0x_2x_3 + 4x_1x_2x_3 - 3x_2^2x_3 + 4x_1x_3^2 - x_2x_3^2 \\ &\quad + 5x_0^2x_4 - 4x_0x_1x_4 + 2x_1^2x_4 + x_0x_2x_4 + 4x_1x_2x_4 - 2x_2^2x_4 \\ &\quad + 4x_0x_3x_4 - 3x_2x_3x_4 - x_0x_4^2 - x_1x_4^2 + 5x_2x_4^2, \end{aligned}$$

Then X is a sextic K3 surface with geometric Picard rank 1.

Proof. We constructed the example so that X_{47} contains the line $V(x_0, x_1, x_2)$. Projecting from this line, and using Theorem 5.1.4, we find that the double cover $X_{47} \to \mathbb{P}^2$ has branch divisor the vanishing of $g_6(u, v, w)$ given by

$$\begin{aligned} &14u^6 + 36u^5v + 40u^4v^2 + 2u^3v^3 + 38u^2v^4 + 40uv^5 + 26v^6 + 7u^5w + 29u^4vw \\ &\quad + 12u^3v^2w + 29u^2v^3w + 2uv^4w + 28v^5w + 29u^4w^2 + 15u^3vw^2 + 12u^2v^2w^2 \\ &\quad + 16uv^3w^2 + 11v^4w^2 + 40u^3w^3 + 31u^2vw^3 + 38uv^2w^3 + 26v^3w^3 + 35u^2w^4 \\ &\qquad + 10uvw^4 + 18v^2w^4 + 2uw^5 + 43vw^5 + w^6 \end{aligned}$$

where $(u : v : w)$ are homogenous coordinates for $\mathbb{P}^2$. Using `Magma`'s algorithm, we computed the Weil polynomial of the double cover model $s^2 = g_6(u, v, w)$ as

$$
\begin{aligned}
(t-47)^2(t^{20} &+ 35t^{19} + 1410t^{18} + 79524t^{17} - 311469t^{16} + 39037448t^{15} + 5504280168t^{14} \\
&- 86233722632t^{13} - 1013246240926t^{12} - 666716026529308t^{11} - 7833913317193690t^{10} \\
&- 1472775702603241372t^{9} - 49443184301680246 06t^{8} - 929531864871588625928t^{7} \\
&+ 1310639929468939962558 48t^{6} + 2053335889501339274674952t^{5} \\
&- 3619004505246110471614602 9t^{4} + 2041118540958806305990636035 6t^{3} \\
&+ 79943809520886580317966578061 0t^{2} + 4383585555395280820768500697011 5t \\
&+ 27666687119623358094507480113424 01)
\end{aligned}
$$

As the Weil polynomial contains no cyclotomic roots besides $(t-47)^2$ coming from the hyperplane section and the line, Theorem 4.5.1 implies that X_p has geometric Picard rank 2.

Next, we use a Gröbner basis calculation on each Schubert cell of the Grassmannian $\mathrm{Gr}(2,5)$ of lines in $\mathbb{P}^4$ to verify that X contains no lines over $\overline{\mathbb{Q}}$, cf. [EJ11]. Finally, we conclude that so $\rho(X) = 1$ by Theorem 5.1.9. $\square$

Remark 5.1.11. Knowing the branch locus $D \subseteq \mathbb{P}^2$ only specifies the polynomial g_6 up to multiplication by a unit. This is insufficient to specify the isomorphism class of the double cover $X \to \mathbb{P}^2$ with branch locus $V(g_6)$, as there can be nontrivial quadratic twists. Explicitly, the double cover models $s^2 = g_6$ and $s^2 = \lambda g_6$ might be different for λ a nonsquare in the base field. In fact, one easily finds examples where these quadratic twists have different point counts over a finite field, hence are be nonisomorphic. Of course, over a finite field of characteristic $\neq 2$, and for a generic choice of g_6, there will be at most one quadratic twist. However, since any two quadratic twists are isomorphic over the algebraic closure, they have the same geometric Picard rank. So, while we do not know which twist gives the double plane model of our original sextic K3 surface, this is not an issue for our argument. After some experimentation, however, we believe that the correct model for the double cover is $s^2 = g_6$, where g_6 is as in Theorem 5.1.4.

Section 5.2

Degree 8 Case

A general polarized K3 surface (X, H) of degree 8 is a complete intersection of three quadrics in $\mathbb{P}^5$ by Theorem 2.4.12. In this case, the space of quadrics $\mathbb{P}^2 = \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{I}_X(2)))$ containing X, called the net of quadrics associated to X, is spanned by the three quadrics. The locus $C \subseteq \mathbb{P}^2$ of degenerate quadrics in the net is a sextic curve. Let $f : Y \to \mathbb{P}^2$ be the discriminant double cover of the net, which is branched over C. The discriminant cover can be defined either in terms of the center of the even Clifford algebra, cf. [Kuz08, §3], [ABB14, §1.6], or the Stein factorization of the relative moduli space of maximal isotropic subspaces, cf. [HVV11, §3], [McK+17, §3.2], associated to the net. These two perspectives are equivalent by [ABB14, Appendix B]. When C is smooth, then Y is smooth and $(Y, f^*\mathcal{O}_{\mathbb{P}^2}(1))$ is a polarized K3 surface of degree 2, called the *discriminant* K3 surface associated to X.

Explicitly, over a field k of characteristic $\neq 2$, if $X = V(q_0, q_1, q_2) \subset \mathbb{P}^5 = \mathbb{P}(V)$ for quadratic forms q_0, q_1, q_2 on V, then we consider the *linear span* quadratic form $q = x_0q_0 + x_1q_1 + x_2q_2$ on $V \otimes_k k[x_0, x_1, x_2]$, see [ABB14, Definition 1.2.3]. The signed discriminant $\operatorname{disc}(q) = -\det(x_0Q_0 + x_1Q_1 + x_2Q_2) \in k[x_0, x_1, x_2]$, with Q_i the symmetric Gram matrix associated to the quadratic form q_i, is then a homogeneous form of degree 6. The discriminant K3 surface $Y \subset \mathbb{P}(1, 1, 1, 3)$ is the variety defined by $y^2 = \operatorname{disc}(q)$ in weighted homogeneous coordinates $(x_0 : x_1 : x_2 : y)$, with the double cover $Y \to \mathbb{P}^2$ given by projection away from the last coordinate, see [McK+17, §5].

Proposition 5.2.1. *The formation of the discriminant K3 surface determines a dominant rational map* $\Phi : \mathcal{K}_8 \dashrightarrow \mathcal{K}_2$.

Proof. The only question is whether the above explicit construction is defined on an open in $\mathcal{K}_8$. The construction is well defined on the locus in $\mathcal{K}_8$ where the polarization H embeds X as a smooth complete intersection of 3 quadrics whose discriminant K3 is smooth. This locus is the complement of finitely many Noether–Lefschetz divisors. Indeed, the failure of smoothness occurs when the polarization fails to be very ample, which occurs on the unigonal locus $\mathcal{K}_{8,1}^0$ or the hyperelliptic locus $\mathcal{K}_{8,2}^0$, see [Sai74, §5, §8]; further, the failure of being a complete intersection occurs on the trigonal locus $\mathcal{K}_{8,3}^0$, see [Sai74, Theorem 7.2]; when X is smooth, the discriminant K3 Y can acquire at most ordinary double points, see [Bea77, §6.2], and this occurs on $\mathcal{K}_{8,1}^{-2}$ (when X contains a line) or $\mathcal{K}_{8,2}^{-2}$ (when X contains a conic). Finally, the rational map is dominant by [Bea77, Proposition 6.23]. $\square$

In particular, there exist Zariski open sets $\mathcal{K}_8^\circ \subset \mathcal{K}_8$ and $\mathcal{K}_2^\circ \subset \mathcal{K}_2$ such that $\Phi : \mathcal{K}_8^\circ \to \mathcal{K}_2^\circ$ is a surjective morphism.

When a complete intersection K3 surface $X \subset \mathbb{P}^5$ of degree 8 is defined over $\mathbb{Q}$, the explicit construction of the discriminant K3 shows that $Y \to \mathbb{P}^2$ is also defined over $\mathbb{Q}$. The following result goes back to Mukai's pioneering work [Muk84; Muk87a; Muk87b] on isogenies between K3 surfaces.

Proposition 5.2.2. *For a smooth complete intersection* $X \subset \mathbb{P}^5$ *of three quadrics over* $\mathbb{Q}$ *whose associated discriminant* $Y \to \mathbb{P}^2$ *is smooth, we have that* $\rho(\overline{X}) = \rho(\overline{Y})$.

Since the base change map $\mathrm{NS}(X \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\overline{\mathbb{Q}}) \to \mathrm{NS}(X \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\mathbb{C})$ is an isomorphism by rigidity for the Néron–Severi group, cf. [MP12, Proposition 3.1], to prove Proposition 5.2.2 we are reduced to working over the complex numbers.

To this end, we briefly review Mukai's theory of isogenies of K3 surfaces and twisted derived equivalence. More details can be found in [Huy16, Section 16.4] and [Huy19]. Recall the Torelli theorem, which states that K3 surfaces X and Y over $\mathbb{C}$ are isomorphic if and only if there exists a Hodge isometry $H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$. More generally, a *Hodge isogeny* between K3 surfaces X and Y over $\mathbb{C}$ is a rational Hodge isometry $H^2(X,\mathbb{Q}) \to H^2(Y,\mathbb{Q})$. Note that any Hodge isogeny restricts to an isomorphism $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H^{1,1}(X,\mathbb{Q}) \to H^{1,1}(Y,\mathbb{Q}) \cong \mathrm{NS}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$, hence X and Y have the same Picard rank. One source of Hodge isogenies is considering moduli spaces of sheaves. The *Mukai lattice* of X is

$$\widetilde{H}(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$$

with the Mukai pairing $v.w = v_2.w_2 - v_0.w_4 - v_4.w_0$, where $v_i.w_j$ is the usual cup product pairing in $H^*(X,\mathbb{Z})$. For $v \in \widetilde{H}(X,\mathbb{Z})$, write $M_X(v)$ for the moduli space of stable sheaves $\mathcal{E}$ on X with Mukai vector

$$v(\mathcal{E}) = \mathrm{ch}(\mathcal{E})\mathrm{Td}_X^{1/2} = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), c_1(\mathcal{E})^2/2 - c_2(\mathcal{E}) + \mathrm{rk}(\mathcal{E})) = v.$$

If $v^2 = 0$ then $M_X(v)$ is also a K3 surface, see [Muk87b, Theorem 1.4]. If $M_X(v)$ is a fine moduli space then X and $M_X(v)$ are *Fourier–Mukai partners*, where the universal sheaf on $X \times M_X(v)$ determines a Fourier–Mukai equivalence $\mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(M_X(v))$ of bounded derived categories. In general, $M_X(v)$ is not a fine moduli space, and X and $M_X(v)$ are *twisted Fourier–Mukai partners*. Indeed, the universal sheaf on $X \times M_X(v)$ is $p^*\alpha$-twisted for the class $\alpha \in \mathrm{Br}(M_X(v))$ that obstructs the fineness of the moduli space, where $p : X \times M_X(v) \to M_X(v)$ is the projection. This universal twisted sheaf determines a twisted Fourier–Mukai equivalence $\mathsf{D}^{\mathrm{b}}(X) \cong \mathsf{D}^{\mathrm{b}}(M_X(v), \alpha)$, see [Căl02, Theorem 1.3]. Following Mukai [Muk87b, Corollary 6.5], Huybrechts [Huy19] constructs a Hodge isogeny between X and $M_X(v)$ using Chern characters of the universal twisted sheaf in $H^2(X, \mathbb{Q}) \otimes H^2(M_X(v), \mathbb{Q})$ to induce a cohomological correspondence.

Returning to the case where $X \subset \mathbb{P}^5$ is a complete intersection of three quadrics with associated discriminant $Y \to \mathbb{P}^2$, Mukai [Muk84, Example 0.9], [Muk87a, Example 2.2] shows that $Y = M_X(v)$ is a moduli space of stable sheaves on X with Mukai vector $v = (2, H, 2)$. Moreover, the universal twisted sheaf on $X \times Y$ induces an equivalence $\mathsf{D}^{\mathrm{b}}(X) \cong \mathsf{D}^{\mathrm{b}}(Y, \alpha)$ for a Brauer class $\alpha \in \mathrm{Br}(Y)[2]$ that can be defined explicitly in terms of the net of quadrics, see [McK+17, Section 3.2, Lemma 15]. This is Mukai's isogeny between a degree 8 K3 surface X and its degree 2 discriminant K3 surface Y, which yields Proposition 5.2.2.

Finally, we give a condition to ensure that a K3 surface of degree 8 over $\mathbb{Q}$ is Noether–Lefschetz general in terms of its discriminant K3 surface.

Proposition 5.2.3. *Let $X \subset \mathbb{P}^5_{\mathbb{Q}}$ be a smooth complete intersection of three quadrics whose associated discriminant $Y \to \mathbb{P}^2$ is smooth, and let $Y_p \to \mathbb{P}^2_{\mathbb{F}_p}$ be the reduction modulo p for a prime $p > 2$ of good reduction of Y. If $\rho(\overline{Y}_p) = 2$, the branch locus of $\overline{Y}_p \to \mathbb{P}^2_{\overline{\mathbb{F}}_p}$ has a tritangent line, and the branch locus of $\overline{Y}$ has no tritangent line, then $\rho(\overline{X}) = \rho(\overline{Y}) = 1$.*

Proof. We follow the strategy outlined in [EJ11, Example 1.7], [HV13, Proposition 5.3], and [McK+17, Proposition 26] to prove that $\rho(\overline{Y}) = 1$, and then appeal to Proposition 5.2.2. Indeed, by Proposition 5.2.2 and the injectivity of the specialization map, we have that $\rho(\overline{X}) = \rho(\overline{Y}) \leq 2$. Assume, to get a contradiction, that

$\rho(\overline{X}) = \rho(\overline{Y}) = 2$. In this case, the image of $\overline{\mathrm{sp}} : \mathrm{Pic}(\overline{Y}) \to \mathrm{Pic}(\overline{Y}_p)$ has rank 2, so has torsion cokernel. By Lemma 2.6.1, the cokernel is torsion free, so it is 0, hence $\overline{\mathrm{sp}}$ is surjective.

Now, the pullback of the tritangent line to $\overline{Y}_p$ splits into two irreducible components. Let L be the divisor class of one of these components, and let $L' \in \mathrm{Pic}(\overline{Y})$ be a lift of L under $\overline{\mathrm{sp}}$. As in [EJ11, Example 1.7], L then must arise from a tritangent line to the branch locus of $Y \to \mathbb{P}^2$, but there are none by hypothesis, giving a contradiction. $\square$

To find explicit X over $\mathbb{Q}$ satisfying the hypotheses of Proposition 5.2.3 for a fixed prime p, one can first iterate over complete intersections $X_p \subset \mathbb{P}^5_{\mathbb{F}_p}$ and check whether the associated discriminant double cover Y_p admits a tritangent. We then use the techniques of Section 4 to check whether $\rho(X_p) = 2$. One can find examples after a number of iterations:

Example 5.2.4. Let $X_{47} = V(q_1, q_2, q_3) \subset \mathbb{P}^5$ over $\mathbb{F}_{47}$, where

$$\begin{aligned}
q_1 = {} & 5x_0^2 + 6x_0x_1 + x_1^2 + 10x_0x_2 + 45x_1x_2 + x_2^2 + 6x_0x_3 \\
& + 45x_1x_3 + 45x_2x_3 + 37x_0x_4 + 39x_1x_4 + 39x_2x_4 \\
& + 2x_4^2 + 10x_0x_5 + 10x_1x_5 + 45x_2x_5 + 8x_4x_5 + 2x_5^2 \\
q_2 = {} & 43x_0^2 + 41x_0x_1 + 41x_0x_2 + 8x_1x_2 + 44x_2^2 + 2x_0x_3 \\
& + 39x_1x_3 + 5x_3^2 + 45x_0x_4 + 2x_1x_4 + 45x_2x_4 + 10x_3x_4 + 3x_4^2 \\
& + 43x_0x_5 + 6x_1x_5 + 2x_2x_5 + 2x_3x_5 + 10x_4x_5 + 3x_5^2 \\
q_3 = {} & 5x_0^2 + 45x_0x_1 + 46x_1^2 + 37x_0x_2 + 2x_1x_2 + 4x_2^2 + 2x_0x_3 \\
& + 8x_1x_3 + 6x_2x_3 + 42x_3^2 + 8x_0x_4 + 39x_1x_4 + 43x_2x_4 + 4x_3x_4 \\
& + 2x_1x_5 + 43x_2x_5 + 43x_3x_5 + 39x_4x_5 + 44x_5^2
\end{aligned}$$

and $(x_0 : \ldots : x_5)$ are homogeneous coordinates on $\mathbb{P}^5$. Then X_{47} is a smooth complete intersection whose discriminant $Y_{47} \to \mathbb{P}^2$ is smooth with branch locus defined by the vanishing of

$$\begin{aligned}13u^6 + 43u^5v + 19u^4v^2 + 7u^3v^3 + 46u^2v^4 + 11uv^5 + 21v^6 + 43u^5w + 22u^4vw \\ + u^3v^2w + 27u^2v^3w + 17uv^4w + 41v^5w + 26u^4w^2 + 42u^3vw^2 + 33u^2v^2w^2 \\ + 46uv^3w^2 + 19v^4w^2 + 42u^3w^3 + 8u^2vw^3 + 34uv^2w^3 \\ + 17v^3w^3 + 41u^2w^4 + 32uvw^4 + 21v^2w^4 + 46uw^5 + 33vw^5 + 17w^6,\end{aligned}$$

where $(u : v : w)$ are homogenous coordinates for $\mathbb{P}^2$. The discriminant Y_{47} admits the tritangent line $V(u+32y+17w)$ and $\rho(\overline{X}_{47}) = \rho(\overline{Y}_{47}) = 2$, which we verify, using Theorem 4.5.1, by computing the Weil polynomial for Y_{47} to be

$$\begin{aligned}(t-47)^2(t^{20} - 15t^{19} - 2491t^{18} + 92778t^{17} + 2387929t^{16} - 34157767t^{15} - 6421660196t^{14} \\ - 53896076645t^{13} + 27357648505002t^{12} + 95245146647044t^{11} - 4924174081652 1748t^{10} \\ + 210396528943320196t^9 + 13349659761453 6664362t^8 - 5809574155447428 91205t^7 \\ - 152907991771376328965156t^6 - 1796668903313671865340583t^5 \\ + 27745701206886846949045 2889t^4 + 2381304964451940690322408 7082t^3 \\ - 141234063486899625228407621 2411t^2 - 18786795237408346374722145844335t \\ + 27666687119623358094507480113 42401).\end{aligned}$$

With such an example in hand, we can choose random lifts X to $\mathbb{Z}$, then using a Gröbner basis algorithm, cf. [EJ08a, Algorithm 8], we can quickly check whether the discriminant double cover Y admits a tritangent line over $\overline{Q}$.

Corollary 5.2.5. *Let* $X = V(q_1, q_2, q_3) \subset \mathbb{P}^5_{\mathbb{Q}}$*, where*

$$\begin{aligned}
q_1 = &-136x_0^2 - 464x_0x_1 - 140x_1^2 - 272x_0x_2 + 374x_1x_2 + x_2^2 + 288x_0x_3 + 186x_1x_3 \\
&+ 468x_2x_3 + 47x_3^2 - 292x_0x_4 - 196x_1x_4 + 274x_2x_4 - 188x_3x_4 + 237x_4^2 - 84x_0x_5 \\
&+ 386x_1x_5 + 562x_2x_5 - 282x_3x_5 - 274x_4x_5 - 139x_5^2 \\
q_2 = &\, 43x_0^2 + 88x_0x_1 + 141x_1^2 - 100x_0x_2 + 384x_1x_2 + 185x_2^2 - 280x_0x_3 - 8x_1x_3 \\
&- 376x_2x_3 - 89x_3^2 + 562x_0x_4 + 190x_1x_4 + 562x_2x_4 + 104x_3x_4 + 144x_4^2 - 98x_0x_5 \\
&- 182x_1x_5 - 468x_2x_5 + 190x_3x_5 - 84x_4x_5 - 44x_5^2 \\
q_3 = &\, 193x_0^2 - 2x_0x_1 + 234x_1^2 - 292x_0x_2 + 190x_1x_2 + 51x_2^2 + 2x_0x_3 - 180x_1x_3 \\
&+ 6x_2x_3 + 183x_3^2 + 8x_0x_4 + 274x_1x_4 + 184x_2x_4 + 286x_3x_4 + 470x_0x_5 - 280x_1x_5 \\
&+ 560x_2x_5 + 90x_3x_5 - 196x_4x_5 + 185x_5^2
\end{aligned}$$

Then $X \subset \mathbb{P}^5$ *is a smooth complete intersection K3 surface of degree 8 over* $\mathbb{Q}$ *with geometric Picard rank 1.*

We note that Elsenhans and Jahnel [EJ08b, §8] give an appealing example of a K3 surface $Y \to \mathbb{P}^2$ defined over $\mathbb{Q}$ with geometric Picard rank 1, whose sextic branch curve $C \subset \mathbb{P}^2$ is the determinant of a 6×6 matrix of linear forms, and whose reductions modulo 5 are smooth and admit a tritangent line. From these examples, and using Proposition 5.2.3, one can also extract K3 surfaces of degree 8 over $\mathbb{Q}$ with geometric Picard rank 1. A similar example can be found in [McK+17, §5.4].

Remark 5.2.6. Note that the locus of polarized K3 surfaces of degree 2 admitting a line tritangent to its branch locus is a Noether–Lefschetz divisor $\mathcal{K}_{2,1}^{-2} \subset \mathcal{K}_2$. In particular, by Proposition 5.2.1, we see that the locus in $\mathcal{K}_8$, consisting of polarized K3 surfaces X of degree 8 whose projective model is a smooth complete intersection of three quadrics in $\mathbb{P}^5$ and whose discriminant $Y \to \mathbb{P}^2$ is smooth and admits a tritangent line, is Zariski closed in $\mathcal{K}_8^\circ$. We will use this in Section 5.3 in the proof that the set of Noether–Lefschetz general K3 surfaces is Zariski dense in $\mathcal{K}_8$.

Section 5.3

Zariski Density

Theorem 5.3.1. *The set of Noether–Lefschetz general K3 surfaces defined over* $\mathbb{Q}$ *is Zariski dense in* $\mathcal{K}_d$ *for* $d \leq 8$.

In this section, we will prove Theorem 1, and along the way streamline the argument that van Luijk employs [Lui07b, Proof of Theorem 1.1] to prove Zariski density in degree 4. Note that, unlike in van Luijk, we are not concerned with whether our Noether–Lefschetz general K3 surfaces have infinitely many $\mathbb{Q}$-points or not, though this could make for an interesting follow up.

In degrees $d \leq 8$, one constructs a subset of $\mathcal{K}_d(\mathbb{Q})$ consisting of Noether–Lefschetz general K3 surfaces. In each case, this subset has the form $T \cap U$, where T is the set of all K3 surfaces with an integral model that reduces to a fixed model mod p, and $U \subset \mathcal{K}_d$ is a Zariski open dense subvariety consisting of the complement of a finite union of Noether–Lefschetz divisors.

We first explain the Zariski open $U \subset \mathcal{K}_d$. For $d = 2$, following Elsenhans and Jahnel [EJ08a; EJ08b], we take $U = \mathcal{K}_d \smallsetminus \mathcal{K}_{2,1}^{-2}$ to be the complement of the tritangent locus; for $d = 4$, following van Luijk [Lui07b], we take $U = \mathcal{K}_4 \smallsetminus \mathcal{K}_{4,1}^{-2}$ to be the complement of the divisor of quartic K3 surfaces containing a line; for $d = 6$, we take $U = \mathcal{K}_d \smallsetminus \mathcal{K}_{6,1}^{-2}$ to be the complement of the divisor of sextic K3 surfaces containing a line; and for $d = 8$, we take U to be the complement in $\mathcal{K}_8$ of the union of $\mathcal{K}_{8,2}^{0}$, $\mathcal{K}_{8,1}^{-2}$, $\mathcal{K}_{8,2}^{-2}$ (which defines the open locus $\mathcal{K}_8^\circ$ where the discriminant K3 is defined and smooth, as in Proposition 5.2.1), as well as the complement of the preimage under

$\mathcal{K}_8^\circ \to \mathcal{K}_2^\circ$ of the tritangent locus $\mathcal{K}_{2,1}^{-2}$.

We now show that T is Zariski dense in $\mathcal{K}_d$. In all cases, this holds by applying the following general result to the unirational varieties $\mathcal{K}_d$ for $d \leq 8$.

Lemma 5.3.2. *Let X be an equidimensional $\mathbb{Z}$-scheme with nonempty generic fiber $X_\mathbb{Q}$ of dimension n and fix a $\mathbb{Q}$-point x of X reducing to an $\mathbb{F}_p$-point x_p of the reduction X_p modulo a prime p. Assume that there is a dominant rational map $f : \mathbb{P}^n_\mathbb{Z} \dashrightarrow X$ such that $x_p = f(z_p)$ for some $\mathbb{F}_p$-point z_p of $\mathbb{P}^n_{\mathbb{F}_p}$. Then the set T of $\mathbb{Q}$-points of $X_\mathbb{Q}$ that reduce to x_p modulo p is Zariski dense in $X_\mathbb{Q}$.*

We believe that this statement should follow from some standard weak approximation type results for unirational varieties, though we could not find an explicit reference. Hence we include a proof below.

Proof. First we prove the result for $X = \mathbb{P}^n_\mathbb{Z}$. Suppose to the contrary that there is a closed proper subvariety $V \subset \mathbb{P}^n_\mathbb{Z}$ whose $\mathbb{Q}$-points contains T. We can assume, without loss of generality, that V is codimension 1, i.e., $V = V(g)$ for some homogenous polynomial g of degree d. For any prime q, the Schwartz–Zippel lemma (a weak form of the Lang–Weil bounds) then implies that the number of $\mathbb{F}_q$-points of V_q is at most dq^{n-1}, an amount that becomes strictly less than $q^n + q^{n-1} + \cdots + 1$ as q increases. Thus, for all but finitely many primes q, there exists at least one $\mathbb{F}_q$-point of $\mathbb{P}^n_{\mathbb{F}_q}$ that is not in V_q.

Fix one such prime $q \neq p$, one such $\mathbb{F}_q$-point y_q of $\mathbb{P}^n_{\mathbb{F}_q}$, and let y be a lift of y_q to a $\mathbb{Q}$-point of $\mathbb{P}^n_\mathbb{Q}$. Now consider the $\mathbb{Q}$-point $qx + py$ of $\mathbb{P}^n_\mathbb{Q}$, where we perform addition in some affine patch $\mathbb{A}^n_\mathbb{Q}$ containing x and y. Since $qx + py$ reduces to x_p modulo p we have that $qx + py \in T$ hence is a $\mathbb{Q}$-point of V. However, this contradicts the fact that $qx + py$ reduces to y_q modulo q, which by construction is not contained in V_q. Hence T must be Zariski dense in $\mathbb{P}^n_\mathbb{Q}$.

Now, assume we have a dominant rational map $f : \mathbb{P}^n_{\mathbb{Z}} \dashrightarrow X$ such that x_p lifts to an $\mathbb{F}_p$-point z_p. By the above argument, the set S of $\mathbb{Q}$-points of $\mathbb{P}^n_{\mathbb{Q}}$ that reduce to z_p modulo p is Zariski dense in $\mathbb{P}^n_{\mathbb{Q}}$. But since f is dominant, the image of a dense subset is dense, so we obtain that $f(S)$ is Zariski dense in $X_{\mathbb{Q}}$. Since $f(z_p) = x_p$ we have that $f(S) \subseteq T$, and hence that T is Zariski dense in $X_{\mathbb{Q}}$ as well. $\square$

Finally, we can give a proof of our main result.

Proof of Theorem 1. For each $d \leq 8$, we construct a prime p and a smooth K3 surface X_p of degree d over $\mathbb{F}_p$ with geometric Picard rank 2. For $d = 2$, this was first done by Elsenhans and Jahnel [EJ08a, Example 28(ii)], [EJ08b, Example 6.1(ii)] in several different ways; for $d = 4$, this was the original case done by van Luijk [Lui07b, § 3]; for $d = 6$, we use the (reduction of the) K3 surface in Corollary 5.1.10; and for $d = 8$, this was first done by Elsenhans and Jahnel [EJ08b, § 8], there is another example in [McK+17, §5.4], and we provide yet another example in Example 5.2.4. In the cases $d \leq 6$, the geometric Picard group is generated by the polarization and a line, whereas in the case $d = 8$, the isogenous discriminant K3 has geometric Picard group generated by the polarization and a line. For $d \leq 6$, taking a lift to $\mathbb{Q}$ that is contained in the open $U = \mathcal{K}_d \smallsetminus \mathcal{K}_{d,1}^{-2}$, i.e., that does not contain a line, forces the lift to have geometric Picard rank 1 by Theorem 5.1.9. For $d = 8$, we take a lift to $\mathbb{Q}$ whose isogenous discriminant K3 is contained in the open $U = \mathcal{K}_2 \smallsetminus \mathcal{K}_{2,1}^{-2}$, i.e., not containing a tritangent line, forcing geometric Picard rank 1 by Proposition 5.2.3.

Lemma 5.3.2 ensures that the set of lifts X to $\mathbb{Q}$ of X_p is Zariski dense in $\mathcal{K}_d$, since for $d \leq 8$, where the general polarized K3 surface is a complete intersection, we have that $\mathcal{K}_d$ admits an integral model that is unirational over $\mathbb{Z}$, given as a quotient of a projective space by a linear algebraic group defined over $\mathbb{Z}$. The intersection of this dense subset of $\mathcal{K}_d(\mathbb{Q})$ with the open subvariety U is still dense in $\mathcal{K}_d$ and consists of polarized K3 surfaces that are Noether–Lefschetz general. $\square$

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