Sample Solutions to Practice Problems for Exam I

Math 11 Fall 2007

October 17, 2008

In these solutions I have shown enough work to get full credit on an exam. Parenthetical comments are extra, for your benefit.

1. TRUE or FALSE: There is a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = y \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2.$$

   Solution: $\boxed{\text{FALSE}}$

   (If there were such a function, then its mixed second partial derivatives would be

$$\frac{\partial^2 f}{\partial y \partial x} = 1 \qquad \frac{\partial^2 f}{\partial x \partial y} = 2x.$$

   These functions are continuous and unequal, but by Clairaut's Theorem, if a function has continuous second partial derivatives then its mixed second partials must be equal.)

2. TRUE or FALSE: There is a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = x \quad \text{and} \quad \frac{\partial f}{\partial y} = y^2.$$

   Solution: $\boxed{\text{TRUE}}$

   (An example is $f(x, y) = \dfrac{x^2}{2} + \dfrac{y^3}{3} + 8$. This is not a straightforward problem for us at this point, although we can check that the mixed partials are equal, so Clairaut's Theorem doesn't rule out such an f. We can hope that this means there is such an f.

To be sure of this, you can look for an f that works. There are two ways you could have gone about this. One is to notice that f_x depends only on x and f_y depends only on y, and guess that therefore f is gotten by adding together two pieces, one depending on x and the other on y. The other is to guess that a function whose partial derivatives are degree-2 polynomials must itself be a degree-3 polynomial,

$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j,$$

where $a \dots j$ are constants, then to find the partial derivatives of such a polynomial, and solve for the constants $a..j$ that give us $f_x = x$ and $f_y = y^2$.)

3. Find the directional derivative of the function

$$f(x, y, z) = 3xy + z^2$$

at the point (1, −2, 2) in the direction from that point toward the origin.

Solution:

Vector from that point toward the origin:

$$\mathbf{v} = \langle -1,\ 2,\ -2 \rangle$$

Unit vector in that direction:

$$\mathbf{u} = \frac{1}{||\mathbf{v}||}\mathbf{v} = \left\langle \frac{-1}{3},\ \frac{2}{3},\ \frac{-2}{3} \right\rangle$$

Directional derivative in direction $\mathbf{u}$

$$D_{\mathbf{u}}f((1,\ -2,\ 2) = \nabla f(1,\ -2,\ 2) \cdot \mathbf{u}$$

$$\nabla f(x,\ y,\ z) = \langle 3y,\ 3x,\ 2z \rangle \qquad \nabla f(1,\ -2,\ 2) = \langle -6,\ 3,\ 4 \rangle$$

$$D_{\mathbf{u}}f((1,\ -2,\ 2) = \nabla f(1,\ -2,\ 2) \cdot \mathbf{u} = \langle -6,\ 3,\ 4 \rangle \cdot \left\langle \frac{-1}{3},\ \frac{2}{3},\ \frac{-2}{3} \right\rangle = \boxed{\frac{4}{3}}$$

4. A skier is on a mountain with equation

$$z = 100 - 0.4x^2 - 0.3y^2,$$

where z denotes height.

(a) The skier is located at the point with xy-coordinates $(1,\, 1)$, and wants to ski downhill along the steepest possible path. In which direction (indicated by a vector $(a,\, b)$ in the xy-plane) should the skier begin skiing?

Solution:

Direction of greatest rate of decrease is opposite of direction of gradient.

$$\nabla g(x,\, y) = \langle -0.8x,\, -0.6y\rangle$$

$$\nabla g(1,\, 1) = \langle -0.8,\, -0.6\rangle \qquad ||\nabla g(1,\, 1)|| = 1$$

Gradient vector is already a unit vector, so unit vector in opposite direction is

$$\mathbf{u} = -\nabla g(1,\, 1) = \boxed{\langle 0.8,\, 0.6\rangle}$$

(b) The skier begins skiing in the direction given by the xy-vector (a, b) you found in part (a), so the skier heads in a direction in space given by the vector $(a,\, b,\, c)$. Find the value of c.

Solution:

The directional derivative in the direction $\mathbf{u}$ (or (a, b)),

$$D_{\mathbf{u}}g(1,\, 1) = \nabla g(1,\, 1) \cdot \mathbf{u} = (-\mathbf{u}) \cdot \mathbf{u} = -1$$

gives the slope. which is the ratio of vertical change to horizontal change. In the direction of the vector $\langle a,\, b,\, c\rangle$, this ratio is $\dfrac{c}{\sqrt{a^2+b^2}}$. So

$$D_{\mathbf{u}}g(1,\, 1) = \frac{c}{\sqrt{a^2+b^2}} = \frac{c}{1} = c.$$

$$c = D_{\mathbf{u}}g(1,\, 1) = \boxed{-1}$$

(c) A hiker located at the same point on the mountain decides to begin hiking downhill in a direction given by a vector in the xy-plane that makes an angle θ with the vector (a, b) you found in part (a). How big should θ be if the hiker wants to head downhill along a path whose slope is at most 0.5 (in absolute value)?

Solution: If the hiker's direction is given by a unit vector $\mathbf{v}$, we want

$$-0.5 \leq D_{\mathbf{v}}(1, 1) \leq 0$$

The angle $\mathbf{v}$ makes with $\mathbf{u}$ (that is, with (a, b)) is the same as the angle $-\mathbf{v}$ makes with $-\mathbf{u}$ (that is, with $\nabla g(1, 1)$). We have

$$D_{\mathbf{v}}(1, 1) = \nabla g(1, 1) \cdot \mathbf{v} = -(\nabla g(1, 1) \cdot (-\mathbf{v})) =$$

$$-(||\nabla g(1, 1)|| \cos\theta) = -\cos\theta$$

So we want

$$-0.5 \leq -\cos\theta \leq 0$$

$$0 \leq \cos\theta \leq 0.5$$

$$\frac{\pi}{2} \geq \boxed{\theta \geq \frac{\pi}{3}}$$

5. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, $\mathbf{u}$ is a unit vector in $\mathbb{R}^3$, and $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ is a differentiable function representing the position of a moving object as a function of time. Let $g(t)$ be the value of f at the object's position at time t. Show that if at time t_0 the object is at position (x_0, y_0, z_0) moving in the direction $\mathbf{u}$ with a speed of 1, then $g'(t_0) = D_{\mathbf{u}}f(x_0, y_0, z_0)$.

   This problem calls for a mathematical argument, not a proof via intuitive physical reasoning.

   Solution:

   Let us set $w = f(x, y, z)$, and write $\mathbf{r}$ in terms of its components as $\mathbf{r}(t) = \langle r_1(t), r_2(t), r_3(t)\rangle = \langle x, y, z\rangle$. We have that w is a function of x, y and z, and x, y and z are functions of t, so w is a function of t ($w = g(t)$) and by the Chain Rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dw}{dt} = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

or rewriting this in terms of the functions and noting where these functions are to be evaluated,

$$g'(t) = \nabla f(r_1(t),\, r_2(t), r_3(t)) \cdot \mathbf{r}'(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$$

When $t = t_0$ we have $\mathbf{r}(t_0) = \langle x_0,\, y_0,\, z_0 \rangle$. Also, $\mathbf{r}'(t_0)$ is the velocity of the moving object at time t_0. The velocity is a vector in the direction of motion (in this case, in the direction of $\mathbf{u}$) whose magnitude is the speed (in this case, 1). Since $\mathbf{u}$ is a unit vector, a vector of length 1 in the direction of $\mathbf{u}$ is just $\mathbf{u}$, and so $\mathbf{r}'(t_0) = \mathbf{u}$. Substituting back into our expression for $g'(t)$, we have

$$g'(t_0) = \nabla f(\mathbf{r}(t_0)) \cdot \mathbf{r}'(t_0) = \nabla f(x_0,\, y_0,\, z_0) \cdot \mathbf{u} = D_{\mathbf{u}} f(x_0,\, y_0,\, z_0)$$

6. Let $f(x, y, z) = x^2 - y^2 + xyz$ and $\mathbf{v} = \langle 3,\, 4,\, 12 \rangle$.

   (a) Find the directional derivative of f in the direction of the vector $\mathbf{v}$ at the point $(1,\, 2,\, -1)$.

   Solution:

   A unit vector in the direction of $\mathbf{v}$ is

   $$\mathbf{u} = \frac{1}{||\mathbf{v}||}\mathbf{v} = \frac{1}{\sqrt{169}} \langle 3,\, 4,\, 12 \rangle = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle$$

   At the point $(1,\, 2,\, -1)$ we have

   $$\nabla f(x,\, y,\, z) = \langle 2x + yz,\, -2y + xz,\, xy \rangle$$

   $$\nabla f(1,\, 2,\, -1) = \langle 0,\, -5,\, 2 \rangle$$

   The directional derivative is

   $$D_{\mathbf{u}} f(1,\, 2,\, -1) = \nabla f(1,\, 2,\, -1) \cdot \mathbf{u} = \langle 0,\, -5,\, 2 \rangle \cdot \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle = \boxed{\frac{4}{13}}$$

   (b) Let $\mathbf{r}(t)$ be a differentiable function giving the position of a moving object as a function of time, such that at time $t = 0$ the object is at the point $(1,\, 2,\, -1)$ moving in the direction of $\mathbf{v}$ at a speed of 1. Compute $\mathbf{r}\,'(0)$.

Solution:

This is the velocity of the object at $t = 0$. The velocity is a vector in the direction of motion (the direction of $\mathbf{v}$) whose length is the speed (1). A vector of length 1 in the direction of $\mathbf{v}$ is the vector $\mathbf{u}$ we computed in part (a):

$$\mathbf{r}'(0) = \boxed{\left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle}$$

(c) Consider the same moving object whose position function is given in part (b). Let $g(t)$ be the value of f at the object's position at time t. Find $g'(0)$.

Solution: By the Chain Rule we have

$$(f \circ \mathbf{r})'(0) = f'(\mathbf{r}(0)) \cdot \mathbf{r}'(0) = \nabla f(\mathbf{r}(0)) \cdot \mathbf{r}'(0) = \nabla f(1, 2, -1) \cdot \mathbf{r}'(0)$$

$$= \langle 0, -5, 2 \rangle \cdot \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle = \boxed{\frac{4}{13}}$$

7. Sometimes a surface in $\mathbb{R}^3$ is easiest to picture by expressing z as a function of polar coordinates r and θ. We may still want to find the partial derivatives of z with respect to x and y, for example, to draw the gradient field.

Give expressions for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in terms of $\dfrac{\partial z}{\partial r}$, $\dfrac{\partial z}{\partial \theta}$, r and θ. Your expressions should be valid when $x > 0$.

Recall that when $x > 0$ rectangular (Euclidean) and polar coordinates are related by the formulas

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan\left(\frac{y}{x}\right),$$

where arctan is the inverse tangent function.

Solution:

By the Chain Rule we have

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y}$$

Find the partials of r and θ:

$$r = (x^2 + y^2)^{\frac{1}{2}}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}2x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}2y = \frac{y}{\sqrt{x^2 + y^2}} = \frac{r\sin\theta}{r} = \sin\theta$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}}\frac{-y}{x^2} = \frac{-y}{x^2 + y^2} = \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}}\frac{1}{x} = \frac{x}{x^2 + y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

Plug in:

$$\boxed{\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cos\theta + \frac{\partial z}{\partial \theta}\frac{-\sin\theta}{r}}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\frac{\cos\theta}{r}}$$

8. Find an equation for the tangent plane to the surface with equation

$$x^2 - y^2 + z^2 = 4$$

at the point $(2,\, 1,\, -1)$.

Solution:

This is a level surface of the function $f(x,\, y,\, z) = x^2 - y^2 + z^2$ so a normal vector to the level surface is $\nabla f(x,\, y,\, z) = \langle 2x,\, -2y,\, 2x\rangle$. A normal vector to the tangent plane is

$$\mathbf{n} = \nabla f(2,\, 1,\, -1) = \langle 4,\, -2,\, -2\rangle$$

and the equation of the plane is

$$\langle 4,\, -2,\, -2\rangle \cdot \langle x - 2,\, y - 1,\, z + 1\rangle = 0$$

$$4(x-2) - 2(y-1) - 2(z+1) = 0$$

$$\boxed{4x - 2y - 2z = 8}$$

9. Find parametric equations for the line through the points

$$A = (1,\ 2,\ 3) \quad \text{and} \quad B = (2,\ 1,\ -1).$$

Solution:

Vector in direction of line:

$$\vec{AB} = \langle 2,\ 1,\ -1\rangle - \langle 1,\ 2,\ 3\rangle = \langle 1,\ -1,\ -4\rangle$$

Vector equation of line:

$$\mathbf{r} = \langle 1,\ 2,\ 3\rangle + t\,\langle 1,\ -1,\ -4\rangle$$

(Parametric) scalar equations:

$$\boxed{x = t+1 \quad y = -t+2 \quad z = -4t+3}$$

10. Find an equation in the form $Ax+By+C = D$ for the plane containing the line

$$\frac{x-1}{2} = y+1 = \frac{z-2}{3}$$

and the point $C = (2,\ 0,\ 3)$.

Solution: Rewrite equation of line:

$$t = \frac{x-1}{2} = y+1 = \frac{z-2}{3}$$

$$\langle x,\ y,\ z\rangle = \langle 2t+1,\ t-1,\ 3t+2\rangle = \langle 1,\ -1,\ 2\rangle + t\,\langle 2,\ 1,\ 3\rangle$$

Vector parallel to line, therefore to plane:

$$\langle 2,\ 1,\ 3\rangle$$

Vector parallel to plane, between points $\langle 1,\ -1,\ 2\rangle$ (on line) and $\langle 2,\ 0,\ 3\rangle$:

$$\langle 2,\ 0,\ 3\rangle - \langle 1,\ -1,\ 2\rangle = \langle 1,\ 1,\ 1\rangle$$

Vector normal to plane:

$$\mathbf{n} = \langle 2,\, 1,\, 3\rangle \times \langle 1,\, 1,\, 1\rangle = \langle -2,\, 1,\, 1\rangle$$

Equation of plane:

$$-2(x-2) + 1(y) + 1(z-3) = 0$$

$$\boxed{-2x + y + z = -1}$$

11. Consider the lines L_1 and L_2 with vector equations

$$\langle x,\, y,\, z\rangle = \langle 1,\, 2,\, 3\rangle + t\,\langle a,\, 1,\, 0\rangle \quad \text{and} \quad \langle x,\, y,\, z\rangle = \langle 2,\, 0,\, 1\rangle + s\,\langle 1,\, 1,\, 0\rangle$$

respectively. Is it possible to choose the constant a so that the lines intersect? (This is not simply a "YES or NO" question. You must explain how you arrived at your conclusion.)

Solution:

Rewrite the equations of the lines:

$$\langle x,\, y,\, z\rangle = \langle 1 + at,\, 2 + t,\, 3\rangle \quad \text{and} \quad \langle x,\, y,\, z\rangle = \langle 2 + s,\, s,\, 1\rangle$$

We see that, whatever the value of a, every point on L_1 has z-coordinate 3 and every point on L_2 has z-coordinate 1. Therefore no point can be on both lines, and the answer to the question is NO: It is not possible to choose the constant a so that the lines intersect.

12. Suppose that $\mathbf{u} \times \mathbf{v} = \langle 5,\, 1,\, 1\rangle$, that $\mathbf{u} \cdot \mathbf{u} = 4$, and that $\mathbf{v} \cdot \mathbf{v} = 9$. Then $|\mathbf{u} \cdot \mathbf{v}|$ is equal to:

(a) 2

(b) 3

(c) 4

(d) 5

(e) None of these.

Solution:

$\boxed{\text{(b)}\ \ 3}$

(From the given information we see that $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta = 3\sqrt{3}$, where θ is the angle between the vectors, $|\mathbf{u}| = 2$, and $|\mathbf{v}| = 3$. We can then figure out that $\sin\theta = \dfrac{\sqrt{3}}{2}$ and $\cos\theta = \pm\dfrac{1}{2}$. From this we see $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta = \pm 3$.)

13. Find the distance between the planes $x + y + z = 1$ and $x + y + z = 4$.

    (a) 2
    (b) $\sqrt{2}$
    (c) $\sqrt{3}$
    (d) 3
    (e) None of these.

Solution:

$\boxed{\text{(c)}\ \ \sqrt{3}}$

(We can take any vector $\mathbf{v}$ between the planes, say the vector $\langle 3, 0, 0\rangle$ between the points $(1, 0, 0)$ and $(4, 0, 0)$, and find its component in the direction of a vector $\mathbf{n} = \langle 1, 1, 1\rangle$ normal to both planes. This component is $\dfrac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{n}|} = \sqrt{3}$.

We could also notice that the line $x = y = z$ is normal to both planes, and intersects them in the points $\left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$ and $\left(\dfrac{4}{3}, \dfrac{4}{3}, \dfrac{4}{3}\right)$. Therefore the distance between the planes is the distance between these two points, or $\sqrt{3}$.)

14. Find the tangent plane to $f(x, y) = x^2 + 2y^2$ at the point $(2, 1, 6)$.

    (a) $x + y + z = 9$
    (b) $4x - 4y - z = -9$
    (c) $4x + 4y - z = 6$
    (d) $4x + 4y + z - 18$

(e) None of these.

Solution:

$\boxed{\text{(c)} \quad 4x + 4y - z = 6}$

15. TRUE or FALSE: The function f is continuous at the point $(0, 0)$, where

$$f(x, y) = \begin{cases} \dfrac{x^4 + y^4}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Solution:

$\boxed{\text{TRUE}}$

(To see this, we must check that

$$\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$$

or that

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + y^4}{x^2 + y^2} = 0$$

We can do this by seeing that

$$0 \leq \frac{x^4 + y^4}{x^2 + y^2} \leq \frac{x^4 + 2x^2y^2 + y^4}{x^2 + y^2} = \frac{(x^2 + y^2)^2}{x^2 + y^2} = x^2 + y^2$$

and then using the Squeeze Theorem.)

16. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are two vectors in $\mathbb{R}^3$ such that if $\mathbf{a}$ and $\mathbf{b}$ are drawn emanating from the origin they both lie in the xy-plane, $\mathbf{a}$ in the third quadrant ($x < 0$ and $y < 0$) and $\mathbf{b}$ in the second quadrant ($x < 0$ and $y > 0$.)

Suppose also that we know $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 2$ and $\mathbf{a} \cdot \mathbf{b} = 1$.

(a) Is the projection of $\mathbf{b}$ onto $\mathbf{a}$ longer than $\mathbf{a}$, shorter than $\mathbf{a}$, or the same length as $\mathbf{a}$?

Solution:

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = 1 \quad |\mathbf{b}| = \sqrt{\mathbf{b}\cdot\mathbf{b}} = 2 \quad \cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{1}{2}$$

The length of the projection of $\mathbf{b}$ onto $\mathbf{a}$ is $|\mathbf{b}|\cos\theta = 1$, so it is the same length as $\mathbf{a}$.

(b) In what direction does $\mathbf{a}\times\mathbf{b}$ point?

Solution:

It must point in a direction normal to both $\mathbf{a}$ and $\mathbf{b}$, that is, normal to the xy-plane, so either the direction given by $\mathbf{k}$ or the direction given by $-\mathbf{k}$. Looking down from the top ($\mathbf{k}$ direction) of the xy-plane we see that from $\mathbf{a}$ to $\mathbf{b}$ is a clockwise direction, so by the right-hand rule, the cross product points in the direction given by $-\mathbf{k}$.

(c) Find the length of $\mathbf{a}\times\mathbf{b}$.

Solution:

If $\cos\theta = \dfrac{1}{2}$ then $\sin\theta = \dfrac{\sqrt{3}}{2}$ (we know it cannot be negative because we always take θ to be an acute angle) so

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta = \boxed{\sqrt{3}}$$

17. A point moves along the intersection of the surface

$$z = x^2 + y^2$$

with the plane

$$x + y = 2$$

from the point $(2, 0, 4)$ to the point $(0, 2, 4)$, in such a way that $x = 2-t$ (where t denotes time and x denotes the x-coordinate of the point's position at time t).

Find the

(a) velocity

(b) acceleration

(c) unit tangent vector $\mathbf{T}$

(d) speed

when the point is at position $(1, 1, 2)$.

Solution:

Parametrize the curve. We are given $x = 2 - t$, and so $y = 2 - x = t$ and $z = x^2 + y^2 = 2t^2 - 4t + 4$. Thus, $\mathbf{r}(t) = \langle 2 - t,\ t,\ 2t^2 - 4t + 4\rangle$, $\mathbf{r}'(t) = \langle -1,\ 1,\ 4t - 4\rangle$, $\mathbf{r}''(t) = \langle 0,\ 0,\ 4\rangle$. At $(1, 1, 2)$, $t = 1$:

velocity $= \mathbf{v} = \mathbf{r}'(1) = \boxed{\langle -1,\ 1,\ 0\rangle}$

acceleration $= \mathbf{a} = \mathbf{r}''(1) = \boxed{\langle 0,\ 0,\ 4\rangle}$

unit tangent vector $= \mathbf{T} = \dfrac{1}{|\mathbf{v}|}\mathbf{v} = \boxed{\left\langle \dfrac{-1}{\sqrt{2}},\ \dfrac{1}{\sqrt{2}},\ 0\right\rangle}$

speed $= |\mathbf{v}| = \boxed{\sqrt{2}}$

18. Approximate $\displaystyle\int_{-0.1}^{0.1} e^{-t^2}\,dt$ using a linear approximation to the function $g(x, y) = \displaystyle\int_x^y e^{-t^2}\,dt$.

Solution:

Using the Fundamental Theorem of Calculus

$$\frac{\partial}{\partial y}\int_x^y e^{-t^2}\,dt = e^{-y^2}$$

$$\frac{\partial}{\partial x}\int_x^y e^{-t^2}\,dt = -e^{-x^2}$$

and near $(x_0,\ y_0) = (0,\ 0)$,

$$g(x, y) \approx g(x_0,\ y_0) + \frac{\partial g}{\partial x}(x_0,\ y_0)(x - x_0) + \frac{\partial g}{\partial y}(x_0,\ y_0)(y - y_0)$$

$$g(x, y) \approx \int_{x_0}^{y_0} e^{-t^2}\,dt + -e^{-x_0^2}(x - x_0) + e^{y_0^2}(y - y_0)$$

$$g(x, y) \approx \int_0^0 e^{-t^2}\,dt + -e^0(x - 0) + e^0(y - 0)$$

$$\int_x^y e^{-t^2}\,dt \approx 0 + (-1)(x) + (1)(y) = y - x$$

$$\int_{-0.1}^{0.1} e^{-t^2}\,dt \approx \boxed{0.2}$$

(This should make intuitive sense. The function e^{-t^2} is not one that we can integrate in a straightforward way, but we know that its value at $t = 0$ is 1. Therefore if we integrate it from -0.1 to 0.1, finding the area under its graph above a small interval around 0, we know the height of its graph on that interval is approximately 1, so the area is approximately 1 times the length of the interval, or in this case approximately 0.2.)

19. Suppose a point moves along the surface $z = f(x, y)$ with its position at time t given by $\vec{r}(t) = (x(t),\, y(t),\, z(t))$. Notice that this means

$$z(t) = f(x(t),\, y(t)).$$

At time t_0 the point is at position $(x_0,\, y_0,\, z_0) = (x_0,\, y_0,\, f(x_0, y_0))$.

(a) Write down an expression for a vector that is normal to the surface $z = f(x, y)$ at the point $(x_0,\, y_0,\, z_0)$. Your expression will involve the partial derivatives of f at $(x_0,\, y_0)$.

Solution:

$$\mathbf{n} = \boxed{\left\langle \frac{\partial f}{\partial x}(x_0,\, y_0),\ \frac{\partial f}{\partial x}(x_0,\, y_0),\ -1 \right\rangle}$$

(This is a formula that you may remember, since we have used it so often, so you can just write it down without explanation— provided you get it right. You can't expect to get partial credit for writing down an incorrect formula, however, so be careful.)

(b) Use the fact that the velocity vector $(x'(t_0),\, y'(t_0),\, z'(t_0))$ is tangent to the surface, and therefore normal to the vector you found in part (a), to solve for $z'(t_0)$ in terms of $x'(t_0)$, $y'(t_0)$, and the partial derivatives of f at $(x_0,\, y_0)$.

Solution:

Since the two vectors are normal, their dot product is zero:

$$
\begin{aligned}
\mathbf{v} \cdot \mathbf{n} &= 0 \\
\langle x'(t_0),\, y'(t_0),\, z'(t_0) \rangle \cdot \left\langle \frac{\partial f}{\partial x}(x_0,\, y_0),\, \frac{\partial f}{\partial x}(x_0,\, y_0),\, -1 \right\rangle &= 0 \\
\frac{\partial f}{\partial x}(x_0,\, y_0)x'(t_0) + \frac{\partial f}{\partial y}(x_0,\, y_0)y'(t_0) - z'(t_0) &= 0 \\
z'(t_0) &= \boxed{\frac{\partial f}{\partial x}(x_0,\, y_0)x'(t_0) + \frac{\partial f}{\partial y}(x_0,\, y_0)y'(t_0)}
\end{aligned}
$$

(c) Now use the chain rule to compute $z'(t_0)$. Your answer should be in terms of $x'(t_0)$, $y'(t_0)$, and the partial derivatives of f at $(x_0,\, y_0)$.

In fact, your answer should be the same as your answer to part (b). You can view parts (a) and (b) as a proof of the chain rule in this case.

Solution:

By the Chain Rule,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

or indicating where each function should be evaluated,

$$\frac{dz}{dt}(t_0) = \frac{\partial f}{\partial x}(x(t_0), y(t_0))\frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x(t_0), y(t_0))\frac{dy}{dt}(t_0)$$

or, since $x(t_0) = x_0$ and $y(t_0) = y_0$),

$$z'(t_0) = \boxed{\frac{\partial f}{\partial x}(x_0,\, y_0)x'(t_0) + \frac{\partial f}{\partial y}(x_0,\, y_0)y'(t_0)}$$

20. (Short answer problem.)

(a) What is the area of the triangle with corners $(0,\, 0,\, 0)$, $(0,\, 1,\, -1)$ and $(1,\, 0,\, 1)$?

Solution:

$\boxed{\dfrac{\sqrt{3}}{2}}$

(The triangle is half of the parallelogram whose edges are vectors from (0, 0, 0) to (0, 1, −1) and to (1, 0, 1), so the triangle has half the area of the parallelogram. The area of the parallelogram is the magnitude of the cross product of those vectors.)

(b) Give an equation for the plane containing (0, 0, 0) and parallel to the plane with equation $3x + 2y - z = 8$.

Solution:

$\boxed{3x + 2y - z = 0}$

(The components of the vector normal to the given plane are the coefficients of x, y and z in its equation $3x+2y-z = 8$. Since the parallel plane has the same normal vector, its equation must have the same coefficients, so its equation has the form $3x+2y-z = D$. You can find the constant term D by plugging in the coordinates of a point on the plane, in this case, $x = y = z = 0$.)

(c) True or False? If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function with continuous second partial derivatives, then $f_{xy} = f_{yx}$.

Solution:

$\boxed{\text{TRUE}}$

(This is Clairaut's Theorem.)

21. A spaceship moves so that its position at time t, for $0 \le t \le 1$, is $\left(t,\ t,\ t^{\frac{3}{2}}\right)$. At time $t = 1$ the engines are turned off, so that the spaceship continues to move at the same velocity it had reached at $t = 1$.

(a) Find the arc length of the path traveled by the spaceship between times $t = 0$ and $t = 1$.

Solution:

For $0 \le t \le 1$,

position $= \left\langle t,\ t,\ t^{\frac{3}{2}} \right\rangle$

velocity $= \left\langle 1,\ 1,\ \dfrac{3}{2}\, t^{\frac{1}{2}} \right\rangle$

speed $= \left|\left\langle 1,\ 1,\ \dfrac{3}{2}\, t^{\frac{1}{2}} \right\rangle\right| = \sqrt{\dfrac{9}{4}t + 2} = \left(\dfrac{9}{4}t + 2\right)^{\frac{1}{2}}$

$$\text{arclength} = \int_0^1 speed\, dt = \int_0^1 \left(\frac{9}{4}t + 2\right)^{\frac{1}{2}} dt$$

$$= \frac{8}{27}\left(\frac{9}{4}t + 2\right)^{\frac{3}{2}}\Bigg|_0^1 = \boxed{\frac{8}{27}\left(\left(\frac{17}{4}\right)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)}$$

(b) Where is the spaceship at time $t = 2$?

Solution:

When $t = 1$ the spaceship is at position $\langle 1,\, 1,\, 1\rangle$ moving with velocity $\left\langle 1,\, 1,\, \frac{3}{2}\right\rangle$. From that point on it moves at constant velocity, so we can parametrize its path by

$$\mathbf{p}(t) = \langle 1,\, 1,\, 1\rangle + (t-1)\left\langle 1,\, 1,\, \frac{3}{2}\right\rangle$$

and when $t = 2$ its position is

$$\mathbf{p}(2) = \langle 1,\, 1,\, 1\rangle + (2-1)\left\langle 1,\, 1,\, \frac{3}{2}\right\rangle = \boxed{\left\langle 2,\, 2,\, \frac{5}{2}\right\rangle}$$

22. S is the surface with equation $z = x^2 + 2xy + 2y$.

(a) Find an equation for the tangent plane to S at the point $(1,\, 2,\, 9)$.

Solution:

The tangent plane to the graph of f has normal vector given by $\left\langle \frac{\partial f}{\partial x},\, \frac{\partial f}{\partial y},\, -1\right\rangle$, or for our function, $\langle 2x + 2y,\, 2x + 2,\, -1\rangle$. At $(1,\, 2,\, 9)$ the normal vector to the tangent plane is $\mathbf{n} = \langle 6,\, 4,\, -2\rangle$, so an equation of the plane is

$$\mathbf{n} \cdot \langle x - 1,\, y - 2,\, z - 9\rangle = 0$$

$$\boxed{6x + 4y - 2z = -4}$$

(b) At what points on S, if any, does S have a horizontal tangent plane?

Solution:

The tangent plane is horizontal when the normal vector is vertical, or when its x and y components, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, are zero. To find these points we set

$$2x + 2y = 0 \quad \text{and} \quad 2x + 2 = 0$$

$$x = -y \quad \text{and} \quad x = -1$$

Also, $z = x^2 + 2xy + 2y$ and so we have

$$\boxed{(x,\, y,\, z) = (-1,\, 1, 1)}$$

23. (a) Show that if $\mathbf{v}$ is any vector function of t and $|\mathbf{v}|$ is constant, then $\mathbf{v}$ is normal (orthogonal, or perpendicular) to $\dfrac{d\mathbf{v}}{dt}$.

Hint: Express $|\mathbf{v}|$ using the dot product, and remember that we have a "dot product rule" for differentiation.

Solution:

$$(|\mathbf{v}|)^2 = \mathbf{v} \cdot \mathbf{v}$$

We differentiate both sides, using on the left the fact that $(|\mathbf{v}|)^2$ is constant, so its derivative is zero, and on the right the dot product rule:

$$0 = \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} + \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 2\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}$$

$$\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = 0$$

These two vectors are normal to each other because their dot product is zero.

(b) Use the result of part (a) to show that if an object travels with constant speed, then its acceleration is normal to its direction of motion.

This agrees with our physical intuition. Acceleration in the direction of motion should correspond to changing speed, and acceleration normal to the direction of motion should correspond to changing direction.

Solution:

If $\mathbf{v}$ is the velocity function and $\mathbf{a}$ the acceleration function, then we know that $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$. Since speed is the magnitude of velocity, we know that $|\mathbf{v}|$ is constant, so by part (a), $\dfrac{d\mathbf{v}}{dt} \perp \mathbf{v}$, or $\mathbf{a} \perp \mathbf{v}$. This means acceleration is normal to velocity, or normal to the direction of motion.

24. (Short answer problem.) Match each of the functions below with the correct pictures of its graph and its level curves. There are pictures on the following pages.

    (a) $f(x, y) = xy$. Graph: $\boxed{\text{C}}$. Level Curves: $\boxed{\text{H}}$.

    (b) $f(x, y) = y - x^2$. Graph: $\boxed{\text{A}}$. Level Curves: $\boxed{\text{G}}$.

    (c) $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$. Graph: $\boxed{\text{B}}$. Level Curves: $\boxed{\text{F}}$.

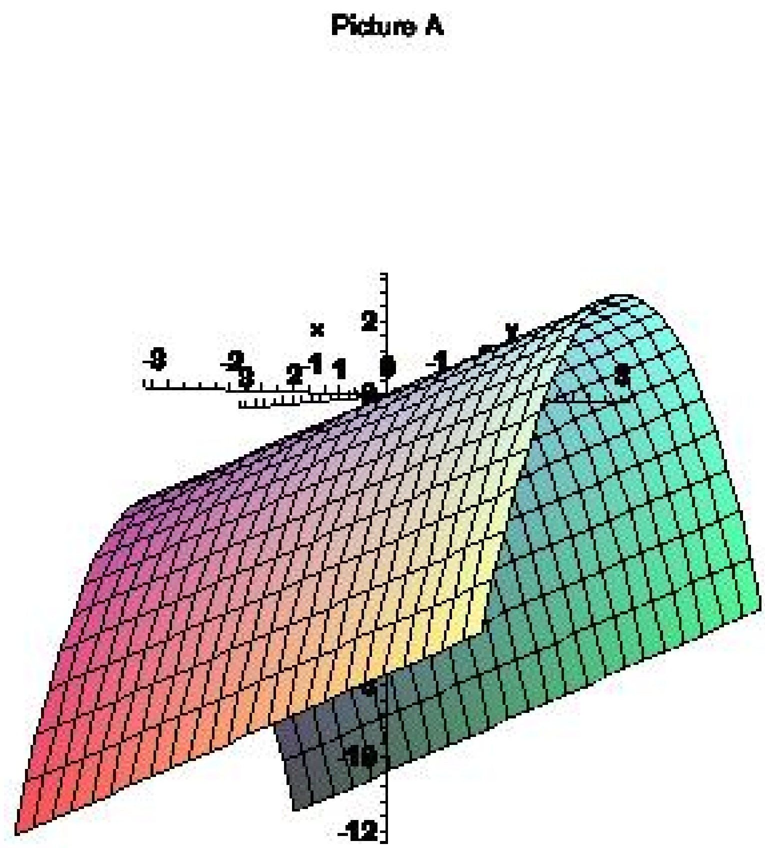

Figure 1: A Picture for Problem 24

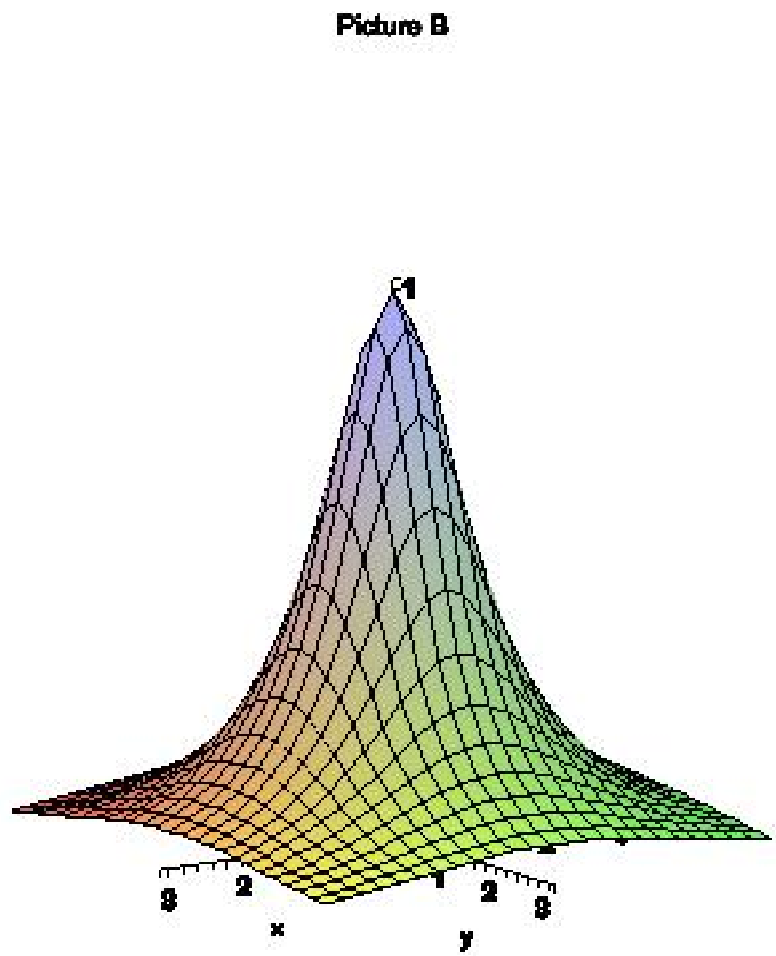

Figure 2: A Picture for Problem 24

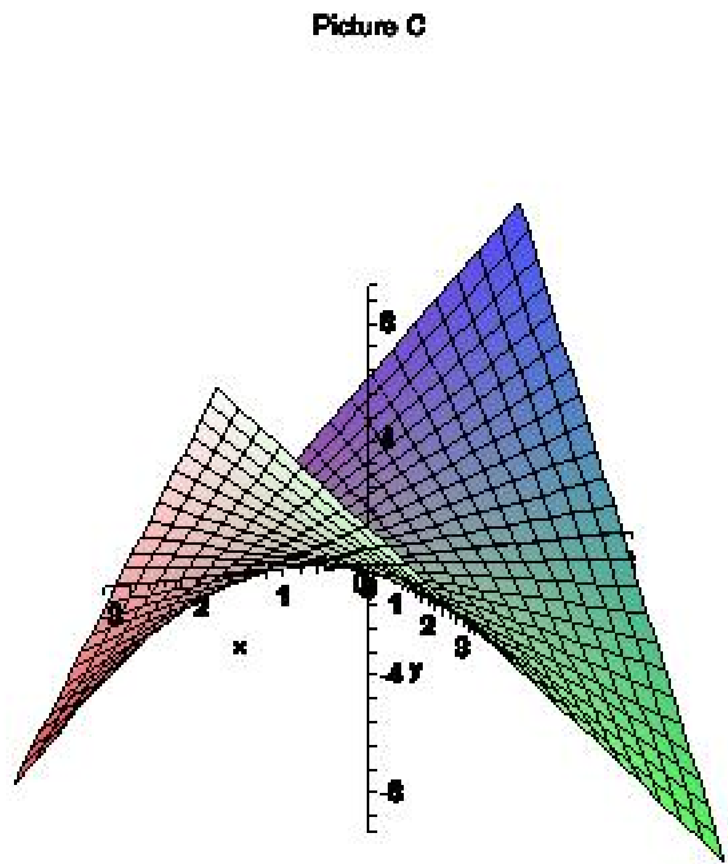

Figure 3: A Picture for Problem 24

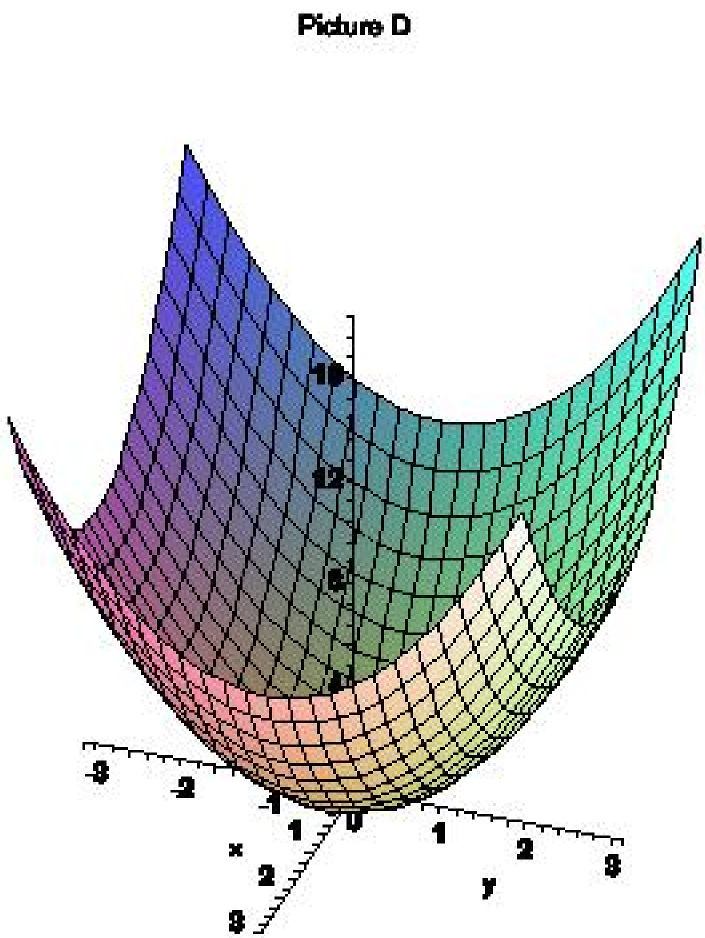

Figure 4: A Picture for Problem 24

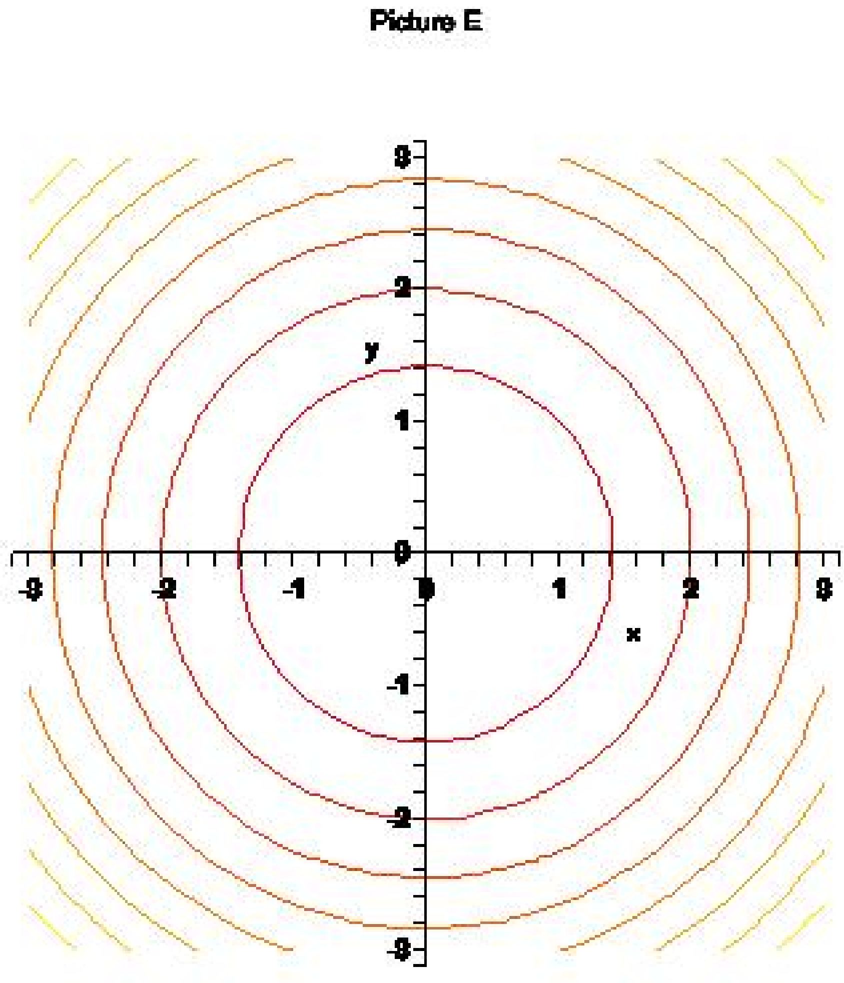

Figure 5: A Picture for Problem 24

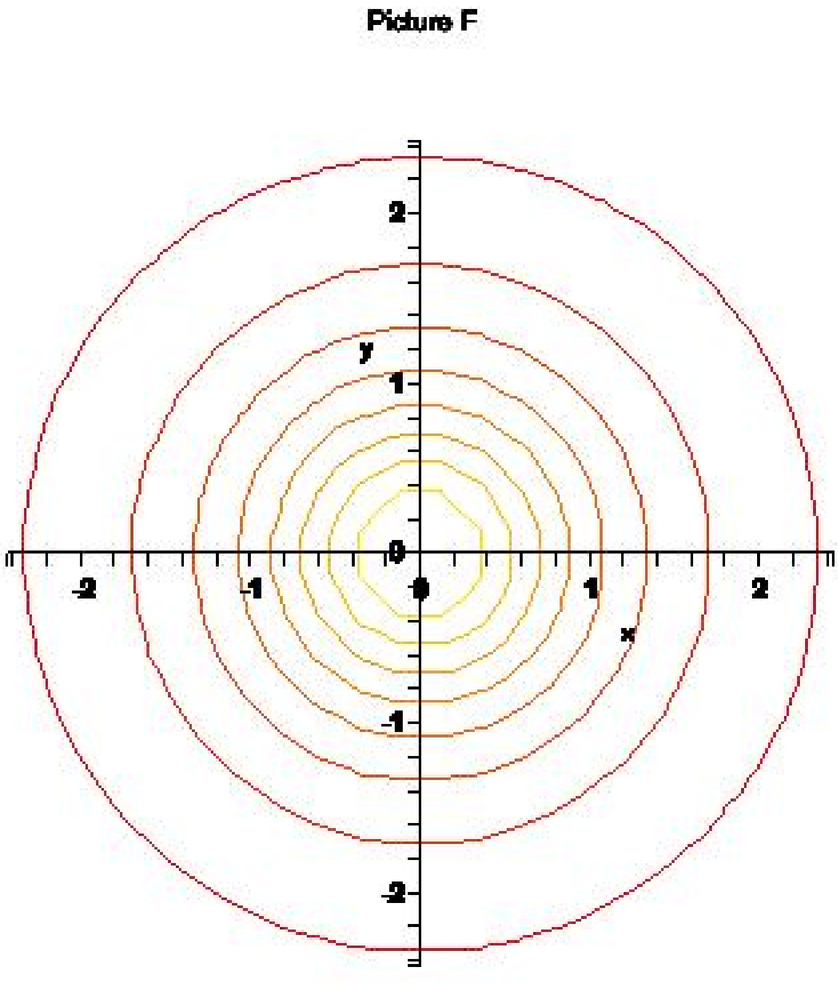

Figure 6: A Picture for Problem 24

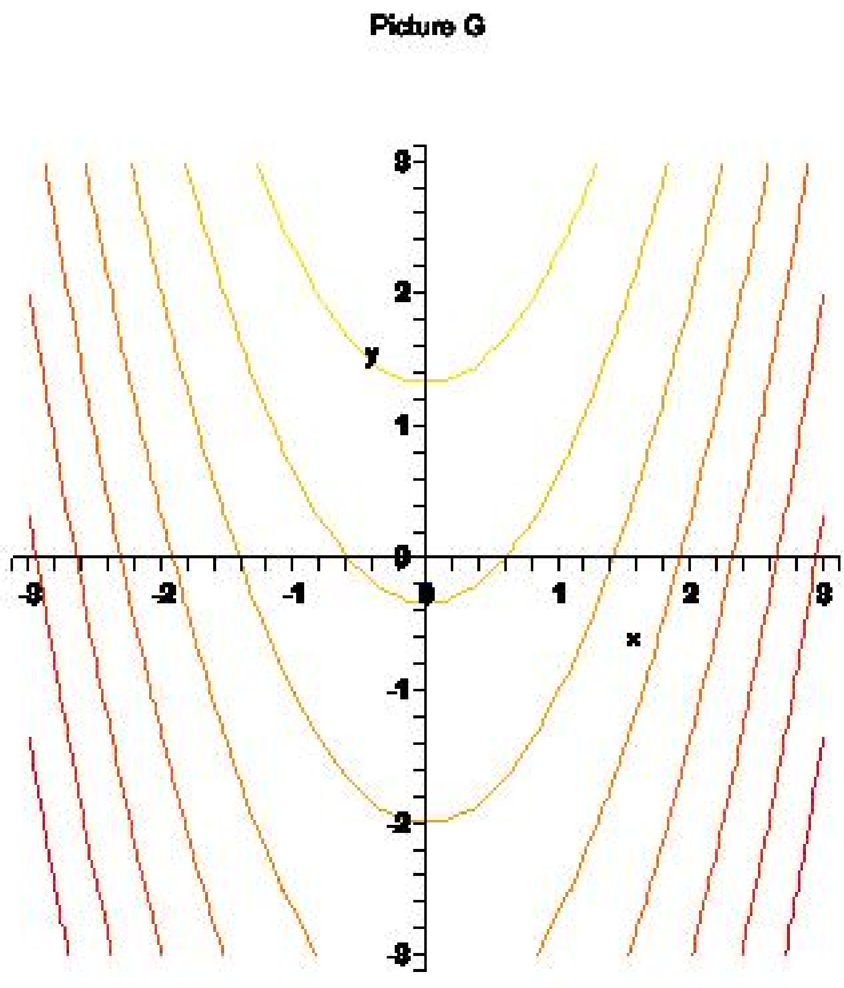

Figure 7: A Picture for Problem 24

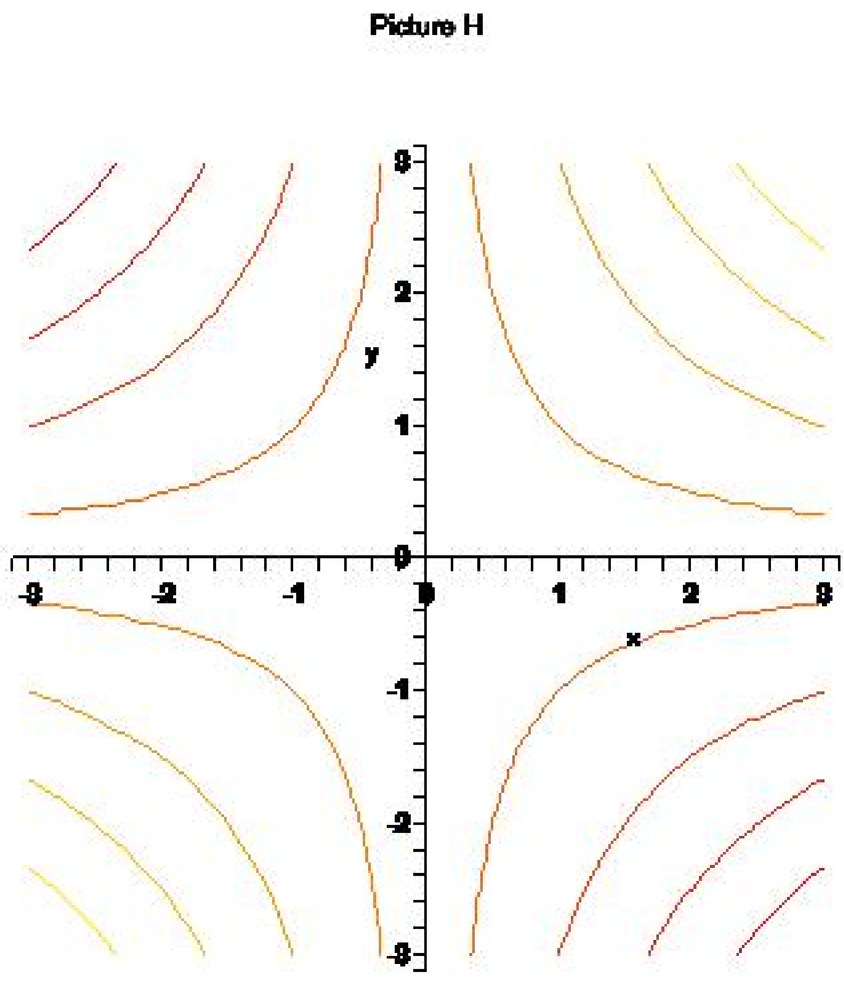

Figure 8: A Picture for Problem 24