Math 11 Midterm 1 Fall 2010 Answer & grading key. (not model solutions, particularly)

1. [10 points] Points $A(0,0,0), B(2,0,0), C(2,2,0), D(0,2,0) E(0,0,2), F(2,0,2), G(2,2,2), H(0,2,2)$ are the eight vertices of a cube.

4 (a) Find an equation for the plane through points A, C and H.

1 for vectors
2 for crossing
1 for plugging in algebra

Find two vectors in plane

cross them

plug in a point, solve for d

$\vec{AC} = 2,2,0$

$\vec{AH} = 0,2,2$

$$\vec{AH} \times \vec{AC} = \begin{array}{ccc} i & j & k \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{array}$$

$4i - 4j + 4k$

$\boxed{4x - 4y + 4z = 0}$

-1 for putting in wrong vector,

-1 for forgetting $-1(j)$ 4x+2+

-2 for dotting instead of plugging in points

2 (b) Calculate the surface area of the triangle ΔACH. (Hint: a triangle is one half of a parallellogram.)

know formula for surface area

find magnitude of cross product from (a)

mult by $\frac{1}{2}$

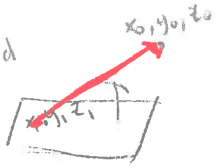
$\frac{1}{2} \; |A \times B| = \frac{1}{2}(4\sqrt{3}) = \boxed{2\sqrt{3}}$

1 for knowing formula
1 for plugging in

4 (c) Calculate the distance from point D to the plane ACH.

$(0,2,0)$

know/derive formula
plug in $\vec{n}$ and D, and d
reduce properly
absolute value



$$\frac{|\vec{b}\cdot\vec{n}|}{|\vec{n}|} = \frac{a(x-x_0)+b(y-y_0)+c(z-z_0)}{\sqrt{a^2+b^2+c^2}} = \frac{ax+by+cz-d}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{4(0)+(-4)(2)+0}{\sqrt{4^2+4^2+4^2}} = \frac{-8}{\sqrt{48}} = \left|\frac{-8}{4\sqrt{3}}\right| = \frac{+2}{\sqrt{3}}$$

because distance

2 for method (memorize/derive)
1 for choosing correct values ($\vec{n}$, P_0)
1 for algebra – absolute value!

if make up some vector $\vec{b}$, −2

~~−1 for putting in~~

if use point D rather than vector from plane to D, −1

if do method correctly, but do a different method as well for wrong answer ⇒ 2 pts

memorized formula wrong, but plugged in correctly: 2/4
–even if use wrong vector in denom.

done correctly w/ wrong answer from a → don't take off

2. [12 points]

[4 pts] (a) Calculate the arc length of the curve

$$\mathbf{r}(t) = \langle 2t\sqrt{t}, \cos t + \sin t, \cos t - \sin t\rangle,$$

Notice we meant $|t|^{3/2}$ here: we apologize (it didn't affect your answers).

for the piece of the curve with $-2/9 \le t \le 2/9$.

[2 pts for method & 2 for accuracy]

velocity $\vec{r}'(t) = \langle 2\cdot\frac{3}{2}t^{1/2}, -\sin t+\cos t, -\sin t - \cos t\rangle$

speed $|\vec{r}'(t)| = \sqrt{3^2(t^{1/2})^2 + (\cos t - \sin t)^2 + (-\cos t - \sin t)^2}$

$= \sqrt{9t + \underbrace{2\cos^2 t + 2\sin^2 t}_{2} + \cancel{2\cos t\sin t} \cancel{- 2\cos t \sin t}}$

$= \sqrt{9t+2}$

$L = \int_{-2/9}^{2/9} \sqrt{9t+2}\,dt = \frac{1}{9}\int_0^4 \sqrt{u}\,du$ via substitution $\begin{cases} u = 9t+2 \\ du = 9\,dt \end{cases}$

$= \frac{1}{9}\cdot\left[\frac{2}{3}u^{3/2}\right]_0^4 = \frac{16}{27}$

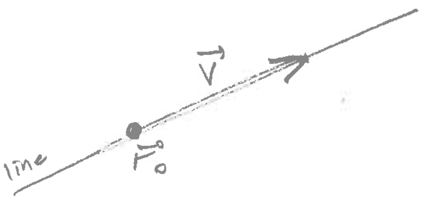
(b) Find a parametric equation for the tangent line to the curve $\mathbf{r}(t) = \langle e^t, e^{2t}, e^{3t} \rangle$ at the point $(2, 4, 8)$. — is $\vec{r_0}$, a point on the line.

(6pts)

$$\vec{r}\,'(t) = \langle e^t, 2e^{2t}, 3e^{3t} \rangle$$

What is t? $\quad \langle 2, 4, 8 \rangle = \langle e^t, e^{2t}, e^{3t} \rangle$ so $t = \ln 2$

$\Rightarrow \quad \vec{v}$ ('velocity' vector for line) $= \vec{r}\,'(\ln 2) = \langle 2, 8, 24 \rangle$



Write parametric eqn for line using a new parameter s, ideally (not confuse w/ t):

$$\vec{r} = \vec{r_0} + \vec{v}s$$

ie $\begin{cases} x = 2 + 2s \\ y = 4 + 8s \\ z = 8 + 24s \end{cases}$ ← fine if called s t here, too

[If didn't introduce a new parameter, -2 pts.]

(c) Calculate the curvature $\kappa(t)$ of the curve

(4pts)

$$\mathbf{r}(t) = \langle \cos(2t), t, \sin(2t) \rangle.$$

Note: for this particular curve, the curvature does **not** depend on the value of t.

Simplest given no other info is $\kappa(t) = \dfrac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$

$$\vec{r}\,'(t) = \langle -2\sin 2t, 1, 2\cos 2t \rangle$$

$$\vec{r}\,''(t) = \langle -4\cos 2t, 0, -4\sin 2t \rangle$$

So $\vec{r}\,' \times \vec{r}\,'' = \langle -4\sin 2t, \underbrace{-8\cos^2 2t - 8\sin^2 2t}_{-8 \text{ by trig.}}, +4\cos 2t \rangle$

$$|\vec{r}\,' \times \vec{r}\,''| = \sqrt{16\sin^2 2t + 64 + 16\cos^2 2t} = \sqrt{80} = 4\sqrt{5}$$

$$|\vec{r}\,'| = \sqrt{4\sin^2 2t + 1 + 4\cos^2 2t} = \sqrt{5}$$

$$\kappa = \frac{4\sqrt{5}}{5^{3/2}} = 4/5$$

as warned, is indep of t.

Full points if evaluated at any fixed t, eg. $t = 0$.

1 point each.

10

3. [10 points] Mark each of the following statements as 'True' of 'False'. If a statement is meaningless, you should mark it as 'False'. (For this question, and for this question only, **you do not have to justify your answers.**)

(a) **True** / **False**. The dot product is distributive, i.e., $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$.

(b) **True** / **False**. The dot product is commutative, i.e., $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ for all vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$.

(c) **True** / **False**. The dot product is associative, i.e., $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$.

(d) **True** / **False**. The cross product is distributive, i.e., $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$.

(e) **True** / **False**. The cross product is commutative, i.e., $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$ for all vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$.

(f) **True** / **False**. The cross product is associative, i.e., $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$.

(g) **True** / **False**. The equation $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b}$ holds for all vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$.

(h) **True** / **False**. The equation $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = \mathbf{a} \times \mathbf{a} + 2\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{b}$ holds for all vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$.

(i) **True** / **False**. The equation $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$ holds for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$.

$a \cdot (b \times c)$

(j) **True** / **False**. The equation $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a}$ holds for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$.

4. [10 points]

3

(a) Either find the limit and prove that it is what you claim, or prove that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^4}$$

$x=0$ gives $\lim_{y\to 0} \dots = 0$.

$y=0$ " "

but, $y = kx$, $\dfrac{xy}{x^2+y^4} = \dfrac{kx^2}{x^2+k^4x^4} = \dfrac{k}{1+k^4x^2}$ ← 1

$\lim_{x\to 0} \dots = k$, depends on k. 1

~~1 point if~~

$\Rightarrow$ limit does not exist ← 1

3

(b) Either find the limit and prove that it is what you claim, or prove that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{\sqrt{x^2+y^2}} = 0$$

proof.: polar coordinates 1

$$\frac{x^2y}{\sqrt{x^2+y^2}} = \frac{r^3\cos^2\theta\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = r^2\cos^2\theta\sin\theta$$

$\lim_{r\to 0} r^2\cos^2\theta\sin\theta = 0$ 1 point.

since this is equiv. to coming in on $y=kx$.

Fixing, over all θ, have: $|r^2\cos^2\theta\sin\theta| \leq r^2$ 1

so this is the max over θ (and $-r^2$ is the min)

$\lim_{r\to 0} r^2 = 0$. 1

$y = kx$

$\dfrac{kx^{2}}{\sqrt{1+k^2}} \xrightarrow{\lim_{x\to 0}} 0$

1

4

(c) Find all points (x, y) where the function f below is continuous. Explain your reasoning. 1

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+2y^2}, & \text{if}(x,y) \neq (0,0) \\ -1, & \text{if } (x,y) = (0,0) \end{cases}$$

- f continuous if $(x,y) \neq (0,0)$ bc. elem. fun
- 3 f NOT cont at $(0,0)$

1 → $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = -1$?

- Try $y = kx$, $f(x,y) = \dfrac{k^3x^3}{x^2+2k^2x^2} = \dfrac{k^3x}{1+2k^2} \xrightarrow{x\to 0} 0$.

It follows that $\lim_{(x,y)\to(0,0)} f(x,y) \neq -1$.

2 for correct proof

Alt. $\left|\dfrac{x^2y}{x^2+2y^2}\right| = \left|\dfrac{r^3\cos^2\theta\sin\theta}{r^2\cos^2\theta + 2r^2\sin^2\theta}\right| = \left|r \cdot \dfrac{\cos^2\theta\sin\theta}{1 + \sin^2\theta}\right| \leq r$

$\lim_{r\to 0} r = 0$.

5. [9 points]

4

(a) Use a linear approximation of $f(x,y) = e^{xy-y^2}$ at the point $(1,1)$ to estimate $f(1.02, 1.01)$.

(+1) Set up $\{ L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

(+1) Derivatives
$$f_x(x,y) = e^{xy-y^2} \cdot y = y e^{xy-y^2}$$
$$f_y(x,y) = e^{xy-y^2} \cdot (x-2y) = (x-2y)e^{xy-y^2}$$

(+1) Plug into f_x, f_y
$$f_x(1,1) = 1 \cdot e^{1-1} = 1 \cdot e^0 = 1$$
$$f_y(1,1) = (1-2)e^0 = -1$$

(+1) Final answer
$$L(1.02, 1.01) = f(1,1) + (1.02-1) - (1.01-1)$$
$$= 1 + 0.02 - 0.01 = \boxed{1.01}$$

2

(b) What is the directional derivative of f at the point $(1,1)$ in the direction of the unit vector $(3/5, 4/5)$?

(+1) Set up
$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$$
$$\nabla f = (f_x, f_y) = (1,-1)$$

(+1) Compute dot product
$$\nabla f \cdot \vec{u} = (1,-1)\cdot(3/5, 4/5)$$
$$= 3/5 - 4/5 = \boxed{-1/5}$$

3

(c) Find an equation for the tangent plane to the surface $x^4 + y^4 + z^4 = 18$ at the point $(1,1,2)$.

$$F(x,y,z) = x^4 + y^4 + z^4 - 18$$

(+1) Derivatives
$$F_x = 4x^3$$
$$F_y = 4y^3$$
$$F_z = 4z^3$$

(+1) normal vector
$$\vec{n} = \langle F_x(1,1,2), F_y(1,1,2), F_z(1,1,2) \rangle$$
$$= \langle 4, 4, 32 \rangle$$

$$4(x-1) + 4(y-1) + 32(z-2) = 0$$

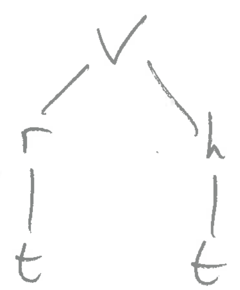
$$4x + 4y + 32z - 4 - 4 - 64 = 0$$

(+1) Final equation
$$\boxed{4x + 4y + 32z = 72}$$
or
$$\boxed{x + y + 8z = 18}$$

6. [6 points] As the in-house mathematician at a landscape gardening firm the following problem arrives on your desk: A manure storage pile is a huge cone of radius 10 feet and height 9 feet. As it is used up, the height decreases by 0.3 feet per day, but due to slippage the radius *increases* by 0.1 feet per day. What is the (initial) rate of manure usage in cubic feet per day?

Useful fact: the volume of a cone $= \frac{1}{3}\pi r^2 h$, where r is the radius and h is the height.

grading scheme:
- 3 = use of chain
- 2 = partials ~~compute~~.
- 1 = plug in



$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= \frac{2\pi}{3} rh \cdot (0.1) + \frac{\pi}{3} r^2 \cdot (-0.3)$$

$$= \frac{2\pi}{3} 10(9)\, 0.1 + \frac{\pi}{3} 10^2 (-0.3)$$

$$= 6\pi - 10\pi$$

$$= -4\pi$$

Rate of usage is 4π ft^3/day.