Math 11 Fall 2010 Midterm 2 Solutions* / grading scheme

1. [8 points] * Note: These are not 'model solutions'.

(a) Find the locations of all local maxima, minima, and saddle points of the function $f(x,y) = (y^2 - x^2)e^y$ in the plane. (Be sure to state the type of each point found.)

$$f_x = -2xe^y$$

$$f_y = 2ye^y + y^2e^y - x^2e^y = \cancel{2y} (2+y)ye^y - x^2e^y$$

$$f_{yy} = +2e^y + 4ye^y + y^2e^y - x^2e^y = (+2+4y+y^2-x^2)e^y$$

$$f_{xx} = -2e^y$$

$$f_{xy} = -2xe^y$$

$\vec{\nabla}f = \vec{0}$: since $e^y \neq 0$, $f_x = 0$ gives $x = 0$

then $f_y = 0$ gives $y = 0$ or -2.

$(0,0)$: $D = -2(+2+4y+y^2)e^{2y} - 4x^2e^y$

$= -4 < 0$ so saddle point.

$(0,-2)$: $D = -2(-2) = +4 > 0$ so local max or min.

Since $f_{xx} < 0$ it's a local max.

(b) [BONUS] Does the function have an absolute maximum or absolute minimum, and why?

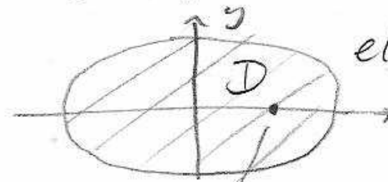
up to +2 available.

Neither. Since $\lim_{y\to+\infty} f(0,y) = \lim_{y\to+\infty} y^2e^y = +\infty$.

$\lim_{x\to\infty} f(x,0) = \lim_{x\to\infty} -x^2e^0 = -\infty$.

2. [10 points]

Find the locations and values of the absolute minimum and absolute maximum of $f(x, y) = xy - y$ over the domain $x^2 + 2y^2 \leq 3$. [Hint: in order to solve the equations you will want to eliminate λ as a first step.]



ellipse.

Interior critical points:

$$f_x = y \xrightarrow{\text{set}} = 0 \qquad y = 0$$
$$f_y = x - 1 \longrightarrow = 0 \qquad x = 1$$

[2pts]

value $f(1,0) = 0$

← this is vital otherwise you don't know it isn't abs max/min.

Boundary extrema: use Lagrange with constraint $g(x,y) = x^2 + 2y^2 = 3$

$$\vec{\nabla} f = \lambda \vec{\nabla} g \quad \text{so} \quad f_x = y = \lambda \cdot 2x \quad ①$$
$$f_y = x - 1 = \lambda \cdot 4y \quad ②$$

[3pts]

Eliminate λ: ① $\Rightarrow$ $\lambda = \frac{y}{2x}$ sub. into ②: $x - 1 = \frac{y}{2x} \cdot 4y$

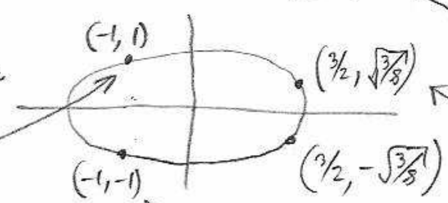
so $x^2 - x = 2y^2 \xrightarrow{\text{use constraint}} = 3 - x^2$

so $2x^2 - x - 3 = 0$

$(2x - 3)(x + 1) = 0 \Rightarrow x = 3/2 \xrightarrow{\text{constraint}} y = \pm\sqrt{\frac{3 - x^2}{2}} = \pm\sqrt{\frac{3}{8}}$

or $x = -1$ " $\rightarrow y = \pm 1$ [4pts]

There are 4 boundary extrema



find values via $f(x,y) = (x-1)y$:

$f = -2$ Abs. min

$f = +2$ Abs max.

$f = \sqrt{\frac{3}{32}}$

$f = -\sqrt{\frac{3}{32}}$

3. [10 points] Let D be the planar domain bounded by $x = 1$ and $x = 2$, $y = 0$ and $xy = 1$.

(a) Write down a form for the above double integral $\iint_D y \sin xy \, dA$ using iterated integrals with only the ordering $dx\,dy$ (i.e. Type II integral).

scheme:

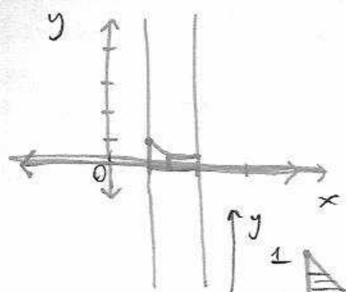
knowing to split into 2: 2

proper first one: 1

proper second one: 1

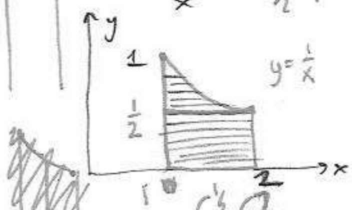
bounds but switched: 1

picture: 1



$y = 1/x$ $\quad z = 1/y \quad y = \frac{1}{2} \quad 1 = 1/y \quad y = 1$

$$\int_{1/2}^{1}\int_{1}^{1/y} y \sin xy \, dx\,dy$$



$$\int_0^{1/2}\int_0^{2} \qquad \int_{1/2}^{1}\int_{1}^{1/y}$$

6

$$\int_0^{1/2}\int_1^2 y\sin xy \, dx\,dy + \int_{1/2}^{1}\int_1^{1/y} y \sin xy \, dx\,dy$$

(b) Evaluate $\iint_D y \sin xy \, dA$

4

2 for first integral

2 for second int

half if evaluate wrong integral correctly

$$\int_0^{1/2}\int_1^2 y\sin xy\,dx\,dy =$$

$$\int_0^{1/2} y \frac{-\cos xy}{y}\Big|_1^2 dy = \int_0^{1/2} -\cos xy \Big|_1^2 = \int_0^{1/2} -\cos(2y) + \cos(y)\, dy$$

$$= -\frac{\sin(2y)}{2} + \sin y \Big|_0^{1/2}$$

$$= -\frac{\sin(1)}{2} + \sin(\tfrac{1}{2})$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$1 - \cos 1 = 2\sin^2(\frac{1}{2})$

$\sin^2(\frac{1}{2}) =$

$\frac{1}{2}(\sin 1 - \cos 1)$

$$\int_{1/2}^{1}\int_1^{1/y} y \sin xy \, dx\,dy$$

$$\int_{1/2}^{1} -\cos xy \Big|_1^{1/y} = \int_{1/2}^{1} -\cos(1) + \cos y = -\cos(1)y + \sin y \Big|_{1/2}^{1} =$$

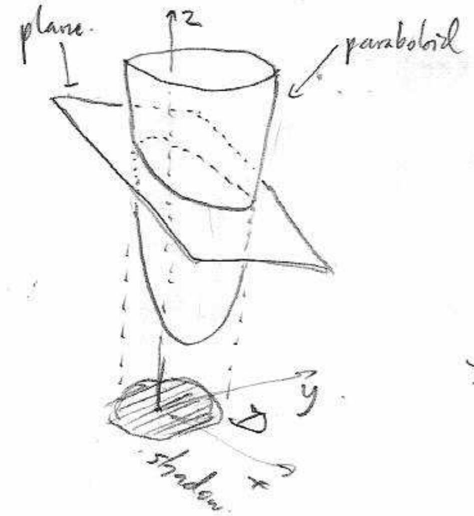
$$-\cos(1) + \sin 1 + \tfrac{1}{2}\cos(1) - \sin(\tfrac{1}{2})$$

$$= -\frac{\sin(1)}{2} - \cos(1) + \sin(1) + \frac{\cos(1)}{2} \qquad \frac{1}{2}\sin(1) - \frac{1}{2}\cos(1)$$

4 each.

4. [12 points] Given below are three solid regions. In each case, analyze the region and write down a corresponding iterated integral. The function $f(x, y, z)$ is unknown in each case, so you only need to write down the correct bounds for the iterated integral.

(a) Let E be the solid region bounded by the paraboloid $z = (x-1)^2 + (y-1)^2$ and the plane $2x + 2y + z = 6$. Write $\iiint_E f(x, y, z)dV$ as an iterated integral of the form



$$\int_{\dots}^{\dots}\int_{\dots}^{\dots}\int_{\dots}^{\dots} \dots \, dz\, dy\, dx.$$

D the x-y plane shadow region: solve intersection

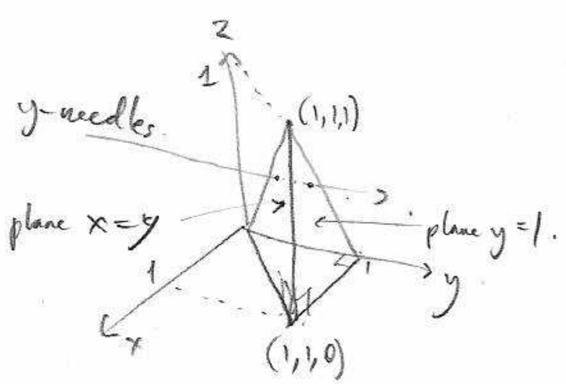
$(x-1)^2 + (y-1)^2 = 6 - 2x - 2y$

$\Rightarrow x^2 + y^2 = 4$, disc radius 2.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{(x-1)^2+(y-1)^2}^{6-2x-2y} \cdots \, dz\, dy\, dx$$

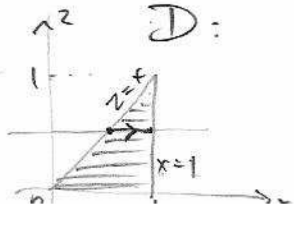
(b) Let E be the tetrahedron ("pyramid") with four vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$ and $(1, 1, 1)$. Write $\iiint_E f(x, y, z)dV$ as an iterated integral of the form

$$\int_{\dots}^{\dots}\int_{\dots}^{\dots}\int_{\dots}^{\dots} \dots \, dy\, dx\, dz.$$



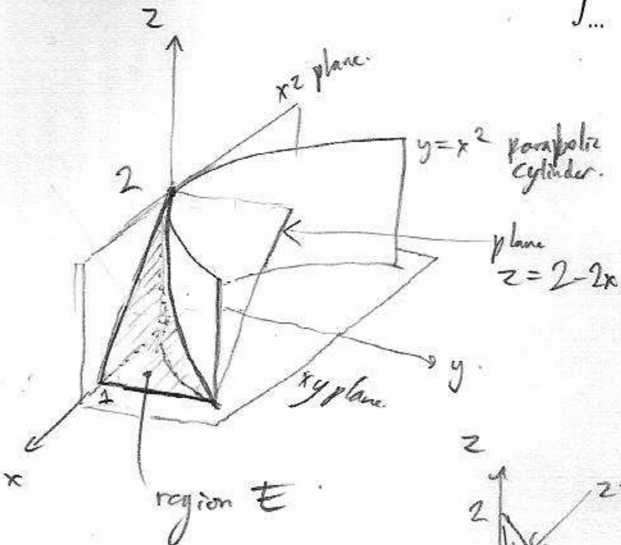
$$\int_{0}^{1}\int_{z}^{1}\int_{x}^{1} \cdots \, dy\, dx\, dz$$

shadow D:

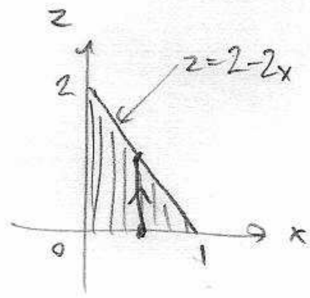


(c) Let E be the solid region bounded by the surfaces $y = x^2$, the plane $2x + z = 2$, the xz-plane, and the xy-plane. Write $\iiint_E f(x, y, z)dV$ as an iterated integral of the form

$$\int_{\cdots}^{\cdots} \int_{\cdots}^{\cdots} \int_{\cdots}^{\cdots} \ldots \; dy\, dz\, dx.$$

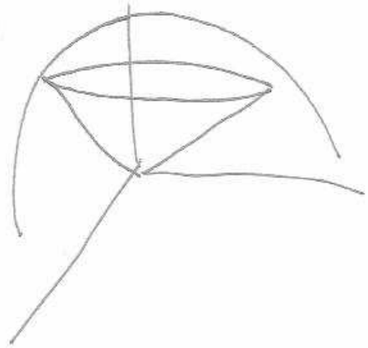


$$\int_0^2 \int_0^{2-2x} \int_0^{x^2} \cdots \; dy\,dz\,dx.$$

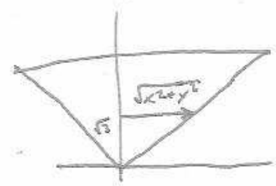


5. (5) Let E be the solid region above the cone $z = \sqrt{3x^2+3y^2}$ and below the sphere $x^2+y^2+z^2=4$. Evaluate the triple integral $\iiint_E z \, dV$.

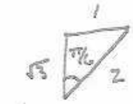
<u>Solution:</u> Use $\boxed{\text{spherical}}$ coordinates. (+1)



Cone: $z = \sqrt{3x^2+3y^2} = \sqrt{3}\sqrt{x^2+y^2}$



$\Rightarrow$



$\phi = \pi/6$

(+2) $\boxed{0 \le \phi \le \pi/6}$

$0 \le \underbrace{x^2+y^2+z^2}_{\rho^2} \le 4 \quad \Rightarrow \quad \boxed{0 \le \rho \le 2}$ (+1)

$\boxed{0 \le \theta \le 2\pi}$ (+1)

bounds (+4)

$z \longmapsto \rho\cos\phi$

$$\iiint_E z \, dV = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} \rho\cos\phi \cdot \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} \rho^3 \sin\phi\cos\phi \; d\rho\, d\phi\, d\theta \quad (+2)$$

$\Downarrow$

$$2\pi \left(\int_0^{\pi/6} \sin\phi\cos\phi \, d\phi\right)\left(\int_0^2 \rho^3 \, d\rho\right) \qquad 2\pi(4)\left(\tfrac{1}{8}\right) = \boxed{\pi} \quad (+1)$$

$$\int_0^2 \rho^3 \, d\rho = \frac{\rho^4}{4}\Big|_0^2 = \frac{16}{4} - 0 = 4 \quad (+1)$$

$$\int_0^{\pi/6} \sin\phi\cos\phi \, d\phi = \int u \, du = \frac{u^2}{2} = \frac{\sin^2\phi}{2}\Big|_0^{\pi/6} = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8} \quad (+1)$$

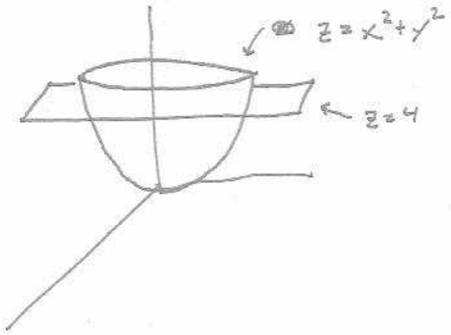
$u = \sin\phi$

$du = \cos\phi \, d\phi$

⑥ Possibly by converting to an appropriate coordinate system, evaluate the iterated integral

$$\int_0^4 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-y^2}} \sqrt{x^2+y^2}\, dx\, dy\, dz$$

Solution:



Region: set enclosed by parabolic cylinder $z = x^2+y^2$, plane $z=4$, and xz-plane.

(+2)

(+1) Use $\boxed{\text{cylindrical}}$ coordinates:

$$z = x^2+y^2 \longmapsto z = r^2$$

1^{st} octant $\Rightarrow \boxed{0 \le \theta \le \pi/2}$ (+1)

$0 \le r^2 \le z \Rightarrow \boxed{0 \le r \le \sqrt{z}}$ (+2)

$\boxed{0 \le z \le 4}$ (+1)

Integral:

$$\int_0^4 \int_0^{\pi/2} \int_0^{\sqrt{z}} r^2\, dr\, d\theta\, dz$$

$$\int_0^{\sqrt{z}} r^2\, dr = \frac{r^3}{3}\Big|_0^{\sqrt{z}} = \frac{z^{3/2}}{3}$$ (+1)

$$\int_0^4 \frac{z^{3/2}}{3} = \frac{z^{5/2}}{3}\cdot\frac{2}{5}\Big|_0^4 = \frac{2(4)^{5/2}}{15} = \frac{2\cdot 32}{15} = \frac{64}{15}$$ (+1)

$$\int_0^{\pi/2} \frac{64}{15}\, d\theta = \frac{64}{15}\cdot\frac{\pi}{2} = \boxed{\frac{32\pi}{15}}$$ (+1)