Math 11 Fall 2010 Final Exam

Tuesday, December 7, 2010

Instructor (circle one): Barnett Van Erp

PRINT NAME: ____________________________________

Instructions: This is a closed book, closed notes exam. You have three hours, do all 11 problems. There are 100 points total, and points for each problem are indicated.

Use of calculators is not permitted.

You must justify all of your answers to receive credit.

The Honor Principle requires that you neither give nor receive any aid on this exam.

GOOD LUCK!

FERPA waiver: By my signature I relinquish my FERPA rights in the following context: This exam paper may be returned en masse with others in the class and I acknowledge that I understand my score may be visible to others. If I choose not to relinquish my FERPA rights, I understand that I will have to present my student ID at my instructors office to retrieve my examination paper.

FERPA waiver signature:

Important hint: In several, but not all, of the problems below, you can simplify the work by applying one of the theorems from Chapter 17. Think before you calculate! If an integral looks impossible, see if you can use the Divergence Theorem, Stokes' Theorem, etc, to replace it with a simpler integral.

You may also find the following well-known identities useful:

$$\sin^2 x = (1 - \cos 2x)/2 \qquad \cos^2 x = (1 + \cos 2x)/2 \qquad \sin 2x = 2 \sin x \cos x$$

1. [8 points] Find an equation for the tangent plane to the surface $x^2z + 2xy^2 + 3yz^2 = 6$ at the point $(x, y, z) = (1, 1, 1)$.

2. [10 points] Find the volume of the intersection of the two solid cylinders $x^2 + y^2 \leq 1$ and $y^2 + z^2 \leq 1$. [*Hint:* This is easiest when evaluated in Cartesian coordinates (x, y, z).]

3. [6 points]

(a) Either find the limit and prove that it is what you claim, or prove that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x}{1+\sqrt{x^2+y^2}}$$

(b) Either find the limit and prove that it is what you claim, or prove that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$$

4. [10 points] Let C be the rectangle that consists of the four oriented straight line segments from $(1,0,0)$ to $(0,0,1)$; from $(0,0,1)$ to $(0,1,1)$; from $(0,1,1)$ to $(1,1,0)$; and finally from $(1,1,0)$ back to $(1,0,0)$. Notice that the closed curve C is *oriented* as indicated, for each of its four pieces in the direction from the first to the second point mentioned above.

(a) Find the value of the line integral

$$\oint_C \sin x\, dx + \ln y\, dy + xyz\, dz.$$

(b) Let C_2 be an *arbitrary* circle in the xy-plane (i.e., with $z = 0$), oriented counter-clockwise. Explain why $\oint_{C_2} \sin x\, dx + \ln y\, dy + xyz\, dz = 0$.

5. [9 points] Let C be the segment of a helix parametrized as $\mathbf{r}(t) = \langle \cos 2t, \sin 2t, t \rangle$, with $0 \leq t \leq \pi$. The curve C is oriented in the direction from $t = 0$ to $t = \pi$. For the vector field

$$\mathbf{F} = \langle \ln x,\ e^{y^2},\ \sin z \rangle,$$

evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

6. [9 points] Let S be the part of the paraboloid $z = x^2 + y^2$ lying under the plane $2y + z = 3$. The surface S is oriented by *downward* pointing normal vectors. Calculate the flux of the vector field

$$\mathbf{F} = \langle 3z - 2x, y - x, z + 2x \rangle$$

across the surface S, i.e., evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

7. [10 points] Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$, and let C denote the curve that is the boundary of S. Finally, let $f(x, y, z) = z^2$.

(a) Evaluate the surface integral $\iint_S f\, dS$.

(b) Evaluate the line integral $\int_C f\, ds$.

8. [9 points] Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F} = \langle x - y, x + y \rangle$$

and C is the curve given by the polar spiral function $r = \theta$ for $0 \leq \theta \leq 2\pi$ followed by the line segment from $(2\pi, 0)$ to the origin, traversed counter-clockwise. [*Hint*: use Green's Theorem]

9. [10 points] Let S be the union of the cylinder $x^2 + y^2 = 1,\ \ 0 \leq z \leq 2$, with normal pointing outwards, with the disc $x^2 + y^2 \leq 1$, $z = 2$, with normal pointing upwards. Let

$$\mathbf{F} = \langle x + z^2, y + e^{z^2}, z + x^2 \rangle$$

be a vector field defined in $\mathbb{R}^3$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

10. [9 points] Let $\mathbf{F}(x, y)$ be the force field $(\frac{1}{y}, 1 - \frac{x}{y^2})$ defined in the upper half plane $y > 0$.

(a) Either find a scalar function f such that $\mathbf{F} = \nabla f$, or else explain why such a function does not exist.

(b) Find the work done by the force field $\mathbf{F}$ in moving a particle along the curve C parametrized by $(1 + t^2, 1 + \sin \pi t)$ starting at $t = 0$ and ending at $t = 1$.

11. [10 points] True or False? On this question (and only on this question) you do not need to show work or explain your answer. (Guessing is allowed and can gain you credit. Therefore, do not skip any items.)

(a) True / False. For any smooth function $f(x, y, z)$ the divergence of ∇f is zero.

(b) True / False. For any smooth function $f(x, y, z)$ the curl of ∇f is zero.

(c) True / False. If S is a *closed* oriented surface in $\mathbb{R}^3$, and $\mathbf{F}$ a smooth vector field, then always $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

(d) True / False. In the conversion of a triple integral from rectangular coordinates (x, y, z) to spherical coordinates (ρ, θ, ϕ) the volume element is transformed as

$$dx\, dy\, dz = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$$

(e) True / False. If C is the circle $x^2+y^2 = r^2$ in $\mathbb{R}^2$ oriented clockwise, then $\int_C 1ds = -2\pi r$.

(f) True / False. If D is a region in the plane $\mathbb{R}^2$ and C is the boundary of D, oriented counter-clockwise, then $\int_C y\, dx$ is equal to the area of D.

(g) True / False. If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$ for which div $\mathbf{F}$ is not zero, then there cannot exist a vector field $\mathbf{G}$ such that $\nabla \times \mathbf{G} = \mathbf{F}$.

(h) True / False. If the domain of a vector field $\mathbf{F}$ is not all of $\mathbb{R}^2$, then $\mathbf{F}$ cannot be conservative.

(i) True / False. $\frac{d}{dt}(|\mathbf{r}|^2) = \mathbf{r}' \cdot \mathbf{r}$ where $\mathbf{r}(t)$ is a position as a function of time.

(j) True / False. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ for all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$.