Math 3, Fall 2009 Midterm 2 Solutions

November 12, 2009

For the multiple choice questions, we omit the choices and just calculate the answer.
1. For the function $f(x) = \ln(e^x + 1)$, what is $f'(x)$?

Solution. Apply the chain rule. We get

$$f'(x) = \frac{1}{e^x + 1} e^x = \frac{e^x}{e^x + 1},$$

which is choice **D**.

2. Which function below (we omit the choices) is an antiderivative for $\ln(x)$?

Solution. An antiderivative of $\ln x$ is a function whose derivative is equal to $\ln x$. Therefore, of the four choices given, you take the derivative of each choice and see if any of the derivatives are equal to $\ln x$. It turns out that $x \ln x - x$ is an antiderivative for $\ln x$, since

$$\frac{d}{dx}(x \ln x - x) = (\ln x + x \cdot (1/x) - 1) = \ln x.$$

Therefore choice **C** is correct.

3. Consider the tangent line to the graph of $y = 2^x$ at $(1, 2)$. The slope is

Solution. the derivative of y at $x = 1$. We know that $y' = (\ln 2)2^x$, so $y'(1) = 2 \ln 2$. This is greater than or equal to 1, since $2 \ln 2 = \ln 4 > \ln e$. (Recall that $e \approx 2.718$, although to solve this problem you only need to know that $e < 4$.) Therefore, the answer is **A**.

4. Consider the curve $x^3 + xy + 2y^3 = 4$ and the point $(1, 1)$ on the curve. The equation of the tangent line to the curve $(1, 1)$ is

Solution. found by calculating the value of y' at $(1, 1)$ and using the point-slope form for the equation of a line. Since we do not have an expression of the form $y = f(x)$ and cannot easily solve for such an expression, we use implicit differentiation. Differentiating both sides of $x^3 + xy + 2y^3 = 4$ with respect to x gives

$$3x^2 + y + xy' + 6y^2y' = 0.$$

Solve for y' in terms of x, y:

$$y' = \frac{-3x^2 - y}{6y^2 + x}.$$

At $(1,1), y'$ has the value $-4/7$. Therefore, the slope of the tangent line through $(1,1)$ is $-4/7$, and the point-slope form for the equation of a line says that this line has equation

$$y - 1 = -\frac{4}{7}(x-1) \Rightarrow y = -\frac{4}{7}x + \frac{11}{7},$$

which is choice **A**.

5. Use the linearization technique to give an approximation to $\sqrt{101}$.

Solution. To apply linearization, we want to find a value of x near 101 at which we can easily evaluate both $f(x) = \sqrt{x}$, and $f'(x) = \frac{1}{2\sqrt{x}}$. The number $x = 100$ clearly is the correct choice. We then have $f(100) = 10, f'(100) = 1/20$, so the slope of the tangent line to $y = f(x)$ through $(100, 10)$ is

$$(y - 10) = \frac{1}{20}(x - 100).$$

We use this tangent line to approximate the value of $f(101) = \sqrt{101}$. At $x = 101$, this line has y-coordinate $y = 1/20 + 10 = 10.05$. Therefore, **B** is the correct answer.

6. Say you are solving the equation $x^3 + x = 1$ with Newton's method, and your initial trial solution, x_0, is 0. What is x_2?

Solution. We will apply Newton's method to $f(x) = x^3 + x - 1$. One has $f'(x) = 3x^2 + 1$, and the equation for Newton's method says

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Apply this equation twice:

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-1}{1} = 1, \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

Therefore, the answer is **B**.

7. Find $\int x^2 + x + 1\,dx$.

Solution. The reverse of the power rule for differentiation tells us

$$\int x^2 + x + 1\,dx = \frac{x^3}{3} + \frac{x^2}{2} + x + C.$$

A careful examination of the choices on the exam shows that this is not equal to any of the first four choices; therefore the answer is **E**, none of the above.

8. Solve the IVP: $\frac{dy}{dx} = x^2y^2, y(1) = 1$.

Solution. Start by separating variables:

$$\frac{dy}{y^2} = x^2\,dx.$$

Integrate both sides:

$$\int \frac{dy}{y^2} = \int x^2\,dx \Rightarrow \frac{-1}{y} = \frac{x^3}{3} + C.$$

Solving for y, and remembering that $+C$ can absorb various constants, we get

$$y = \frac{3}{C - x^3}.$$

To solve for C, we use the initial condition $y(1) = 1$. This implies

$$1 = \frac{3}{C - 1} \Rightarrow C = 4.$$

Therefore, the solution to the IVP is given by

$$y = \frac{3}{4 - x^3},$$

which is choice **D**.

9. Give the general solution to the differential equation $\frac{dy}{dx} = y + 1$.

Solution. We solve this differential equation by separating the variables, again:

$$\frac{dy}{y + 1} = dx.$$

Integrate both sides:

$$\int \frac{dy}{y+1} = \int dx \Rightarrow \ln|y+1| = x + C.$$

Raise both sides to the e:

$$|y+1| = e^{x+C} = Ce^x.$$

We can remove the absolute value sign because any potential negative sign is absorbed by the C term. Doing that and then solving for y gives

$$y = Ce^x - 1,$$

so choice **A** is the correct answer.

10. A radioactive substance decays so that the rate of decay is proportional to the amount present. After 4 years, exactly one-third of the radioactive substance remains. The half-life of the substance is

Solution. **C**, more than 2 years, but less than 4 years. First notice that if the half-life of a substance is short, it decays quickly. If a substance has half-life 2 years, then after 4 years only 1/4 of it would remain. Therefore, the substance in the question must have a half-life greater than 2 years. On the other hand, if a substance has a half-life of 4 years, then after 4 years half of the substance would remain. Therefore, the substance in the question has a half life of less than 4 years.

11. For the function $f(x) = x^3 - 3x + 1$, which of the following holds?

Solution. **D**, concave down on $(-\infty, 0]$ and concave up on $[0, \infty)$. We can check that this is correct by calculating $f'(x) = 3x^2 - 3, f''(x) = 6x$, and observing that $f''(x) < 0$ exactly when $x < 0$, and $f''(x) > 0$ exactly when $x > 0$. One can determine that all the other choices are wrong, either by examining the sign of $f'(x)$ or $f''(x)$.

12. A $10'$ ladder is leaning against a wall but sliding down, with the bottom of the ladder moving from the wall at $t + 1$ feet per second at time t. At time 2 seconds, the base is $8'$ from the wall. How fast is the top of the ladder falling at this instant?

Solution. This is a related rate problem. Let the distance of the base of the ladder from the wall be x, and let the distance of the top of the ladder from the ground be y. Then $x^2 + y^2 = 10^2$, by the Pythagorean Theorem. Furthermore, we know that $\frac{dx}{dt} = t + 1, x(2) = 8$, and $y(2) = 6$. (We find $y(2)$ from the Pythagorean Theorem). Take the derivative of $x^2 + y^2 = 10^2$ with respect to t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

We want to know what dy/dt is at time $t = 2$, so we plug in the values of x, y, and dx/dt at $t = 2$:

$$2(8)(3) + 2(6)\frac{dy}{dt}(2) = 0 \Rightarrow \frac{dy}{dt}(2) = -4.$$

Therefore, the ladder is sliding at a rate of 4 feet per second, and the correct answer is **B**.

13. Use Euler's method with step size 1 to approximate $y(3)$ given $y(-1) = 0$ and $\frac{dy}{dx} = x + 2y$.

Solution. Recall that the formula for Euler's method is

$$y_{n+1} = y_n + \frac{dy}{dx}(y_n) \cdot h$$

where h is the step size. In this problem, $h = 1$. We start with the point $(x_0, y_0) = (-1, 0)$, so

$$\begin{aligned} x_1 = 0, y_1 &= 0 + (-1)(1) = -1 \\ x_2 = 1, y_2 &= -1 + (0 + 2(-1)) = -3 \\ x_3 = 2, y_3 &= -3 + (1 + 2(-3)) = -8 \\ x_4 = 3, y_4 &= -8 + (2 + 2(-8)) = -22 \end{aligned}$$

y_4 corresponds to the y-coordinate of the estimate with x-coordinate equal to 3, so we estimate $y(3)$ to be -22.

14. Sketch the graph of $y = x^3 + 3x^2 - 2$, indicating in a side statement the intervals where the function is increasing, where it is decreasing, where it is concave up, and where it is concave down.

We begin by calculating the first and second derivatives of y:

$$\begin{aligned} y' &= 3x^2 + 6x = 3x(x + 2), \\ y'' &= 6x + 6 = 6(x + 1) \end{aligned}$$

Let us consider the sign of y' first. Notice that when $x < -2$, $y' > 0$, since then $x < 0, x + 2 < 0$, so their product is > 0. When $-2 < x < 0$, $y' < 0$, because $x < 0$ while $x + 2 > 0$. Finally, when $x > 0$, $y' > 0$ as well. Therefore, y is increasing on $(-\infty, -2]$, and $[2, \infty)$, and decreasing when x on $[0, 2]$. The points corresponding to $x = -2, 0$ are critical points, and correspond to a local maximum and local minimum, respectively, as we can already see by examining how the sign of y' changes as x passes these points.

For concavity, we examine the sign of y''. Here, $y'' < 0$ when $x < -1$, and $y'' > 0$ when $x > -1$. Therefore, y'' is concave down when $x < -1$, and concave up when $x > -1$.

One can then draw a sketch of $y = x^3 + 3x^2 - 2$ using all this information. In particular, the point $(-2, 2)$ is a local maximum, the point $(-1, 0)$ is an inflection point and a root of $y = x^3 + 3x^2 - 2$, and the point $(0, -2)$ is a local minimum. As $x \to -\infty, y \to -\infty$, and as $x \to \infty, y \to \infty$. Furthermore, one should note that $y(1) = 2$, so the function increases quite quickly as we pass $x = 0$.