Deformations of Chaotic Billiards and a New 'Wall Formula' for Heating Rate

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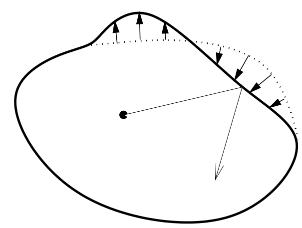
Funding: National Science Foundation, ITAMP

Summary of the thesis

1. Dissipation rate in deforming chaotic billiards

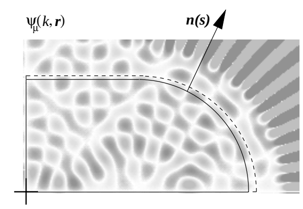
Doron Cohen PRL **85**, 1412 (2000); `nlin.CD/0003018`; `nlin.CD/0008040`



- Quantum-classical correspondence (QCC)
- 'Special' deformations
- Improving the 'wall formula'

2. Improved numerical methods for billiard quantization

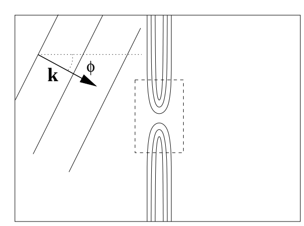
Michael Haggerty unpublished



- New 'sweep' methods for eigenstates
- Analysis of scaling method of Vergini and Saraceno
- Explained quasi-orthogonality on the boundary

3. Mesoscopic QPC conductance, and scattering in the half-plane

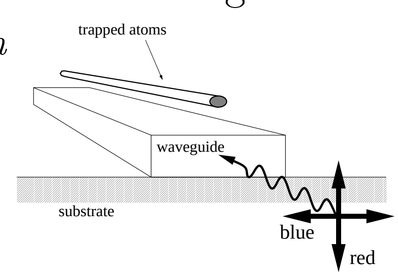
Miriam Blaauboer, Areez Mody `cond-mat/0008279`



- Transmission cross-section replaces Landauer
- Derivation of maximum tunneling conductance

4. Design of atom waveguide using two-color evanescent light fields

Steve Smith, Maxim Ol'shanii, Kent Johnson, Allan Adams, Mara Prentiss PRA **61**, 023608 (2000)

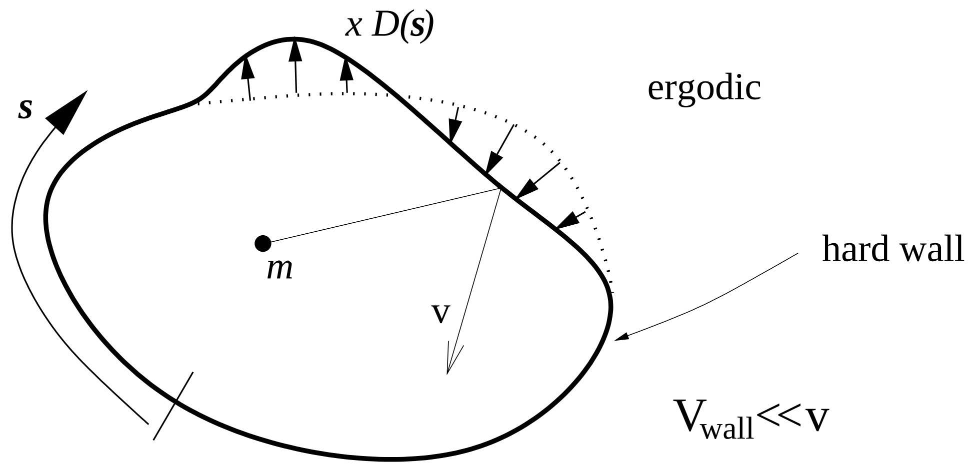


- Solved arbitrary dielectric optical bound modes
- Design equations for atom guiding properties

Outline of today's talk

- Deforming billiards + motivation
- Key statements :
  1. Special class of deformations
  2. Vergini's numerical method
  3. Improve 'wall formula'
- Theory of heating (classical picture)
- The 'wall formula'
- Explain 'special' deformations
- Quantum-classical correspondence + quasi-orthogonality
- Components of a general deformation
- Improved 'wall formula' in action

Deforming billiard (cavity) systems



$D(\mathbf{s})$ = deformation shape function
$x(t) = A \sin \omega t$ periodic 'driving'

Question: At what rate is the 'gas' particle heated up?

Motivations

- Dissipation rate of vibrations of nuclei (3D)
  - never considered ω-dependence
- Driven mesoscopic 2D quantum dots (*e.g.* x = gate voltage)
  - find heating rate of electrons

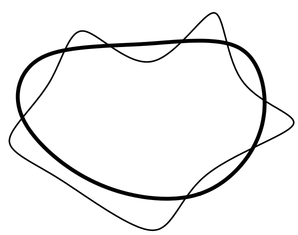
1) Special class of deformations

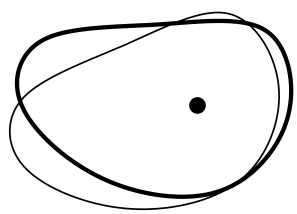
How does heating rate depend on deformation $D(\mathbf{s})$?

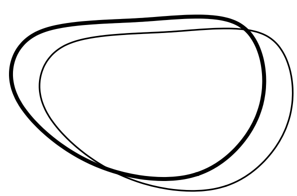
$$\text{heating} \quad \frac{d}{dt}\langle\mathcal{H}\rangle \;=\; \mu(\omega)\cdot\frac{1}{2}(A\omega)^2$$

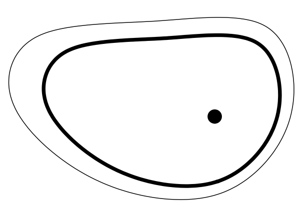
$$\mu(\omega) \;=\; \text{friction coefficient}$$

For low frequency $\omega \ll$ collision rate:

Generic  $\mu(\omega) \sim \text{const}$

Rotation  $\mu(\omega) \sim \omega^2$

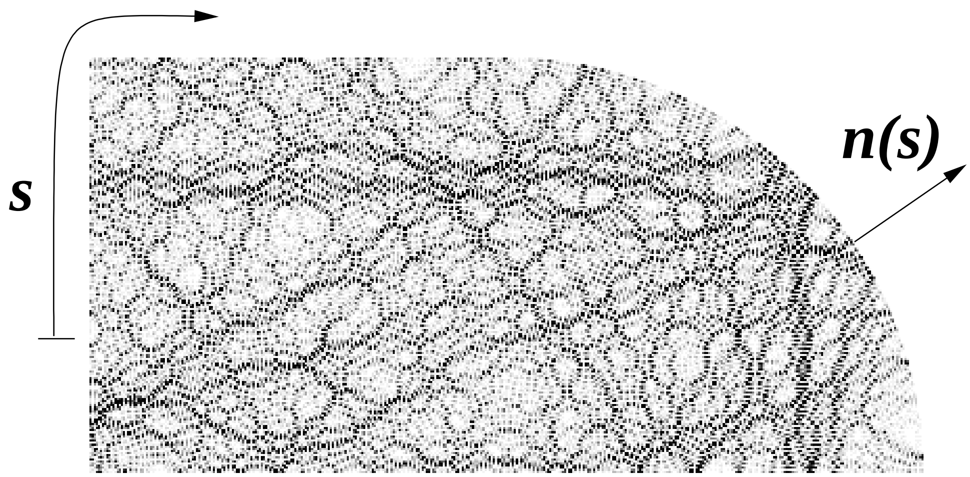
Translation  $\mu(\omega) \sim \omega^4$

Dilation  $\mu(\omega) \sim \omega^4$

Does **not** depend on billiard shape or chaoticity

Special class surprise: **friction vanishes at dc** $\mu(\omega \to 0) = 0$

2) Vergini's numerical method



eigenstate ψ_n

boundary function $\varphi_n \equiv \mathbf{n} \cdot \nabla \psi_n$

Quasi-orthogonality on boundary:

$$\oint (\mathbf{r} \cdot \mathbf{n}) ds \, \varphi_n(\mathbf{s}) \varphi_m(\mathbf{s}) \quad \propto \quad \delta_{nm} \; + \; error\left(\frac{E_n - E_m}{\hbar}\right)$$

Numerical method for finding eigenstates ψ_n (Vergini)

- 10^3 times more efficient than any other known method!
- finds clusters of eigenstates simultaneously
- needs *error* small close to diagonal

No-one has known size of *error*—including Vergini

I show: mean square $error(\omega) \;=\; a\,\omega^4$

Due to $\mu(\omega) \sim \omega^4$ for **dilation** deformation

3) Improved 'wall formula' estimate for $\mu(0)$

Nuclear physics interest (last 25 years):

- estimate friction $\mu(0)$ given $D(\mathbf{s})$
- assume uncorrelated collisions (strong chaos)
  $\rightarrow$ 'wall formula'
- they knew $\mu(0) = 0$ for translations and rotations
  $\rightarrow$ *ad hoc* corrections

But I know special class of $D(\mathbf{s})$ for which $\mu(0) = 0$

(even for strong chaos)

I show: there is consistent way to *subtract* all special components of a general $D(\mathbf{s})$

...**now** applying wall formula gives *improved* estimate of $\mu(0)$.

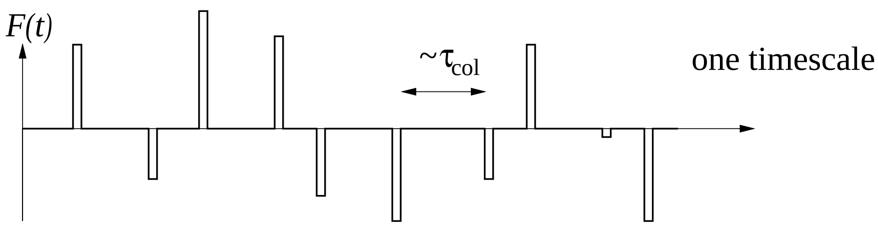
This

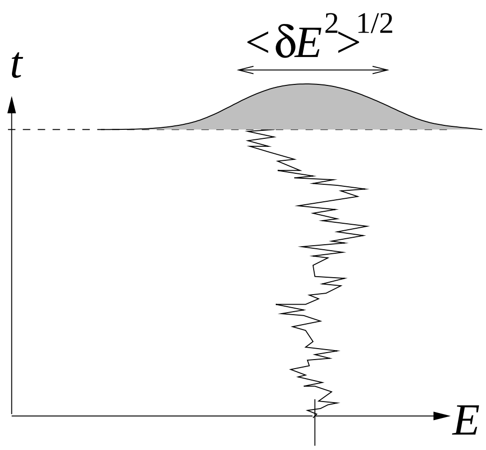
- replaces all *ad hoc* corrections
- incorporates special nature of dilation for first time
- allows predictions for $\mu(\omega)$ at finite ω for first time

Theory of heating rate: energy spreading

Particle energy gets random ‘kicks’: $\dot{\mathcal{H}} = -\dot{x}(t)\mathcal{F}(t)$

where generalized ‘force’ on parameter $\mathcal{F}(t) \equiv -\frac{\partial \mathcal{H}}{\partial x}(t)$

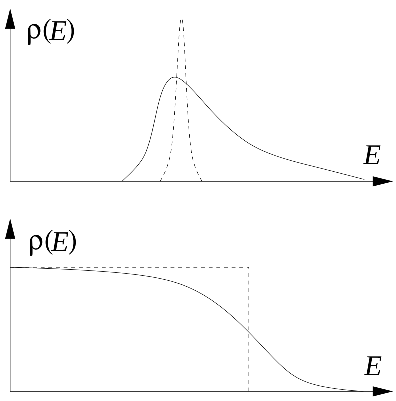




Diffusive spreading $\langle \delta E^2 \rangle \approx 2 D_{\mathrm{E}} t$

$$D_{\mathrm{E}} = \frac{1}{2}\tilde{C}_{\mathrm{E}}(\omega) \cdot \frac{1}{2}(\omega A)^2$$

$\tilde{C}_{\mathrm{E}}(\omega) \equiv$ **power spectrum** of $\mathcal{F}(t)$



Causes irreversible energy growth
(Jarzynski, Cohen)

Why?

- D_{E} increases with E
- drift due to Liouville’s theorem

Friction coefficient $\mu(\omega) \propto \tilde{C}_{\mathrm{E}}(\omega)$ ...relation depends on $\rho(E)$

The 'wall formula': white noise approximation

Seek simple analytic **estimate** of $\tilde{C}_{\mathrm{E}}(0)$

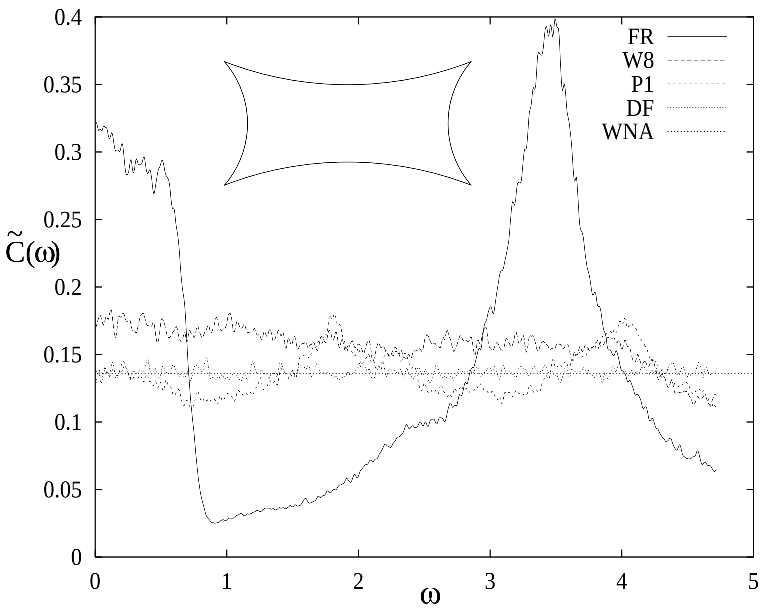
Assume uncorrelated collisions (impulses): $\mathcal{F}(t) = $ white noise

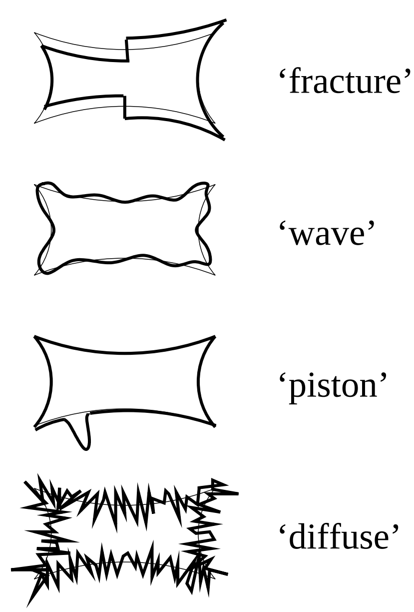
$$\tilde{C}_{\mathrm{E}}(0) \quad \approx \quad \left\langle \sum_{\text{impulses}} \text{self-correlation of impulses} \right\rangle_{\mathrm{E}}$$

$$\xrightarrow{\text{ergodicity}} \quad b_{\mathrm{E}} \cdot \oint [D(\mathbf{s})]^2 d\mathbf{s}, \qquad \text{'wall formula' (Swiatecki)}$$

Predicts power spectrum **flat**: $\tilde{C}_{\mathrm{E}}(\omega) = $ const

Numerical tests in 2D billiard: $D(\mathbf{s})$





If $D(\mathbf{s})$ emphasizes correlations → deviates from WNA

Explanation of 'special' deformations

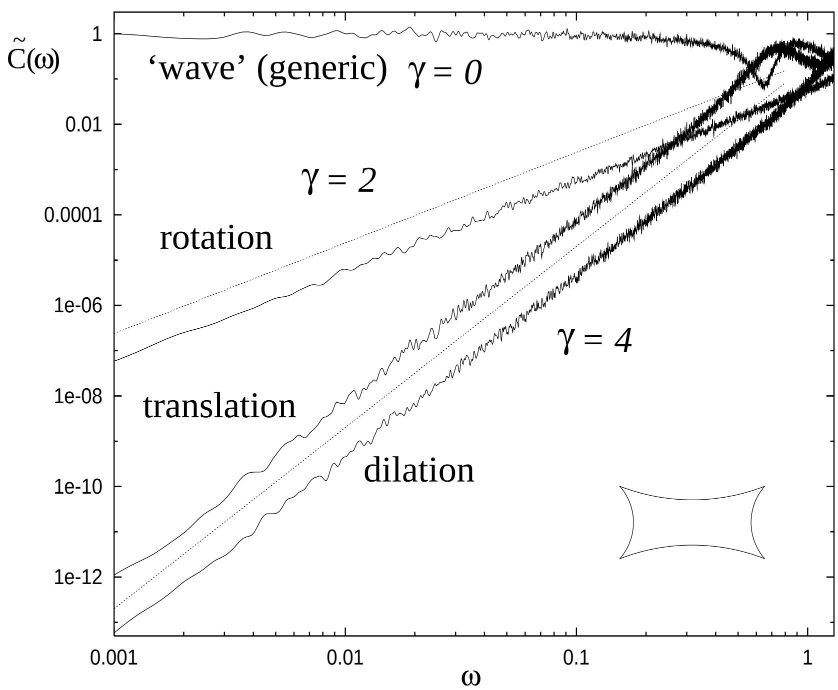
Some $D(\mathbf{s})$, WNA **fails**: $\tilde{C}_{\rm E}(0) = 0$ (even in strong chaos)

Could always write $\mathcal{F}(t) \;=\; \left(\frac{d}{dt}\right)^n \mathcal{G}(t)$

$\Rightarrow$ power spectra $\tilde{C}_{\rm E}(\omega) \;=\; \omega^{2n}\tilde{C}_{\mathcal{G}}(\omega)$ $\quad (\, d/dt \xrightarrow{\rm FT} i\omega \,)$

Special deformations: $\mathcal{G}(t) \;=\;$ some function of $(\mathbf{r}(t), \mathbf{p}(t))$

$\Rightarrow$ $\tilde{C}_{\mathcal{G}}(0)$ finite $\quad \Rightarrow$ $\tilde{C}_{\rm E}(0)$ vanishes



Generic $\tilde{C}_{\mathcal{G}}(\omega) \sim \omega^0 \;\rightarrow$ power laws $\tilde{C}_{\rm E}(\omega) \sim \omega^{\gamma}, \quad \gamma = 2n$

- Dilation: $(n=2)$ $\mathcal{G}(t) = -\frac{1}{2}m\mathbf{r}^2$ since $\mathcal{H} = $ const
- Translation: $(n=2)$ $\mathcal{G}(t) = m\mathbf{e}\cdot\mathbf{r}$ $\mathbf{e} = $ const direction
- Rotation: $(n=1)$ $\mathcal{G}(t) = -\mathbf{e}\cdot(\mathbf{r}\times\mathbf{p})$ ang. mom.

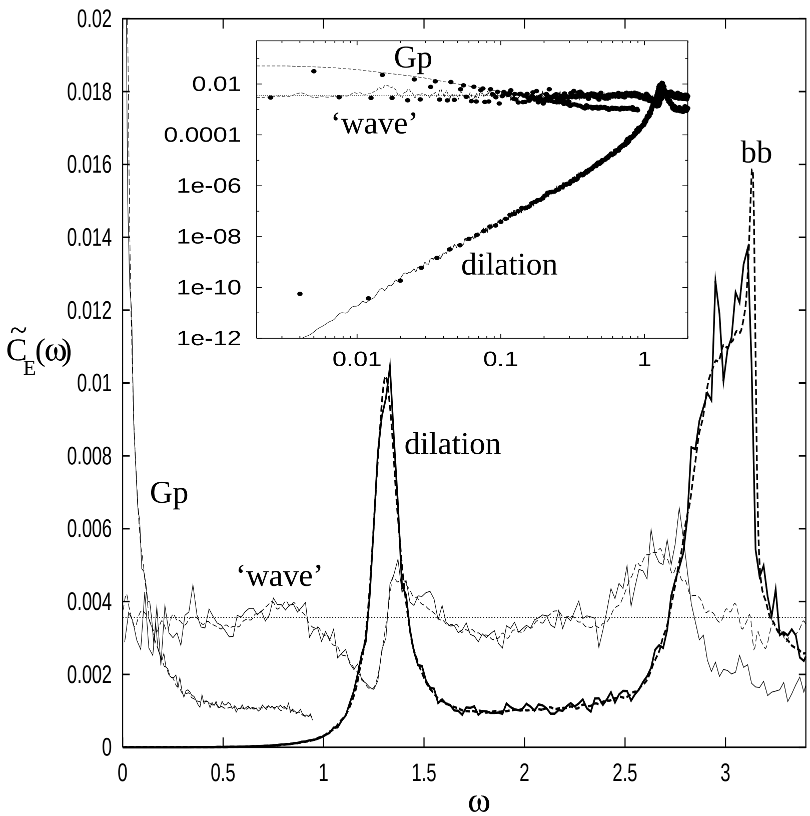
Quantum-classical correspondence (QCC) ?

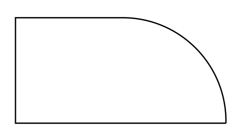
$$\tilde{C}_{\mathrm{E}}(\omega) \underset{(E\text{-shell})}{\overset{\substack{\text{classical} \\ \text{average}}}{\longleftarrow}} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \mathcal{F}(t)\mathcal{F}(t+\tau) \underset{(\text{many states at } E)}{\overset{\substack{\text{QM} \\ \text{expectation}}}{\longrightarrow}} \tilde{C}_{\mathrm{E}}^{\mathrm{qm}}(\omega)$$

Semiclassical prediction: $\tilde{C}_{\mathrm{E}}^{\mathrm{qm}}(\omega) \approx \tilde{C}_{\mathrm{E}}(\omega)$ (Feingold, Wilkinson)

Why? Correspondence of dynamics up to ergodic time

Numerical test—excellent agreement (even for special):



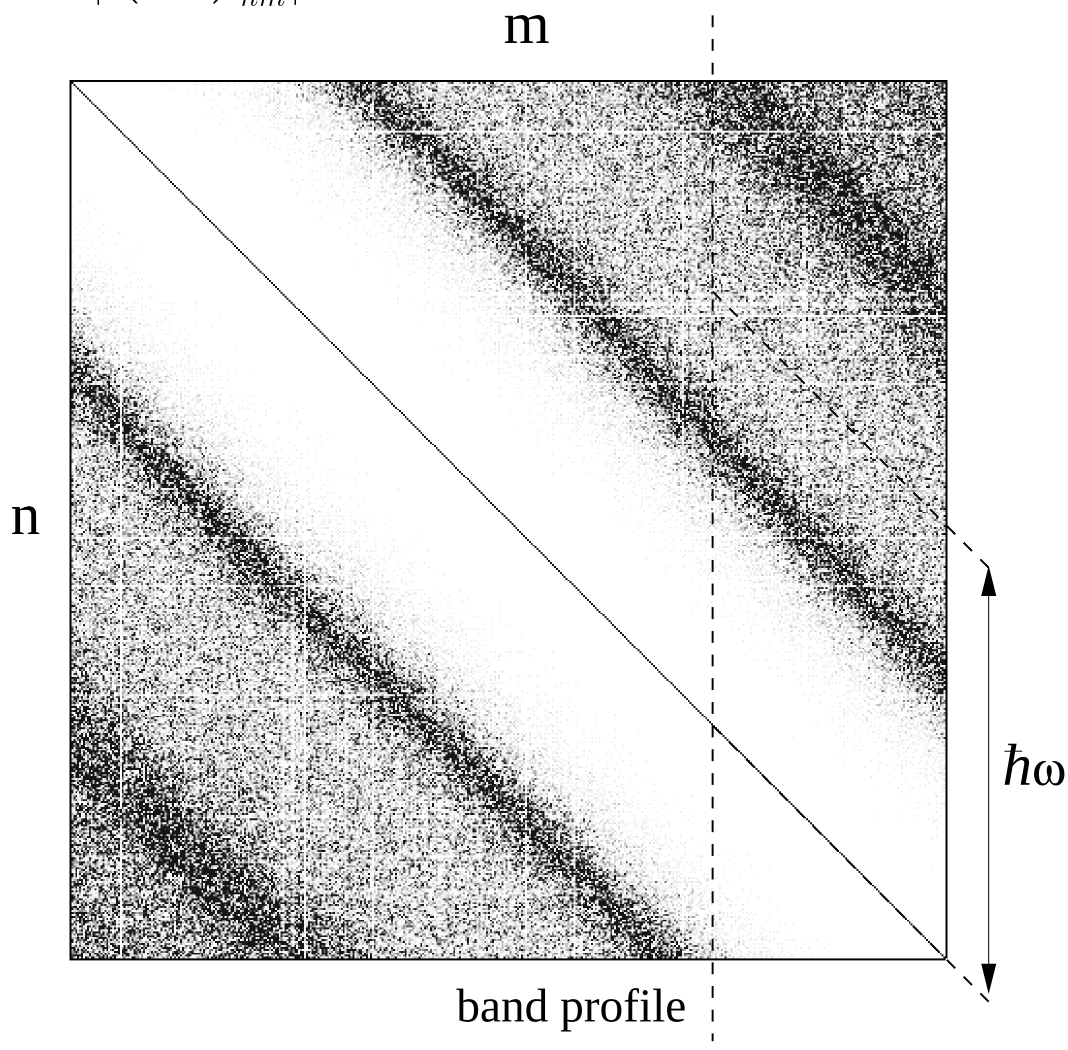


⇒ Equivalence: QM (linear response) = classical heating rate

Quasi-orthogonality on the boundary

$\tilde{C}_{\mathrm{E}}^{\mathrm{qm}}(\omega)$ = average 'band profile' of $\left(\frac{\partial \mathcal{H}}{\partial x}\right)_{nm}$ ↙eigenstate basis

Matrix $\left|\left(\frac{\partial \mathcal{H}}{\partial x}\right)_{nm}\right|^2$ for **Dilation** $(D(\mathbf{s}) = \mathbf{r}\cdot\mathbf{n})$:



$$\left(\frac{\partial \mathcal{H}}{\partial x}\right)_{nm} \propto \oint (\mathbf{r}\cdot\mathbf{n}) d\mathbf{s}\, \varphi_n(\mathbf{s})\varphi_m(\mathbf{s}) \quad \propto \quad \delta_{nm} + O(\omega_{nm}^2)$$

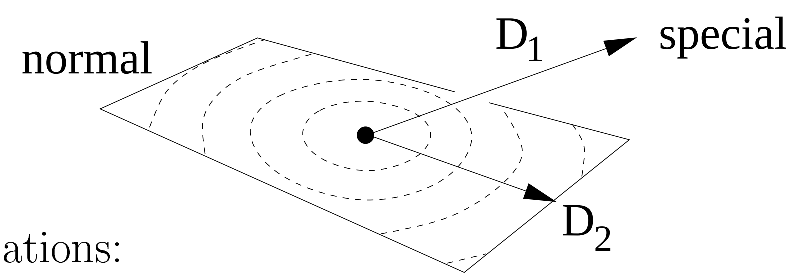
Semiclassical QCC ⇒ Success of scaling method (Vergini and Saraceno)

General deformations: 'special' and 'normal' components

Want to 'subtract' special part of $D(\mathbf{s})$, only then apply WNA for $\tilde{C}_{\rm E}(0)$

- Special defs = linear sub-space (*special* + *special* = *special*)
- 'Normal' (WNA-good) defs = linear sub-space (ergodicity)

Show sub-spaces orthogonal. In what sense?



Add deformations:

$$D(\mathbf{s}) = D_1(\mathbf{s}) + D_2(\mathbf{s}) \longrightarrow \tilde{C}(\omega) = \tilde{C}_1(\omega) + \tilde{C}_2(\omega) + \overbrace{2\tilde{C}_{1,2}(\omega)}^{\text{cross-correlation}}$$

$D_1(\mathbf{s})$	$D_2(\mathbf{s})$	result (analytic, numerically verified)
special	general	$\tilde{C}_{1,2}(\omega \to 0) = 0$
general	normal	$\tilde{C}_{1,2}(\omega) \approx b_{\rm E} \cdot \oint d\mathbf{s}\, D_1(\mathbf{s}) D_2(\mathbf{s})$

$$\Rightarrow \text{ Orthogonality : } \quad 1 \perp 2 \Leftrightarrow \oint d\mathbf{s} D_1(\mathbf{s}) D_2(\mathbf{s}) = 0$$

Defines inner product: now can decompose any $D(\mathbf{s})$ into special and 'normal' ($\perp$ special) components...

Improved estimate for $\tilde{C}_{\mathrm{E}}(0)$ in action

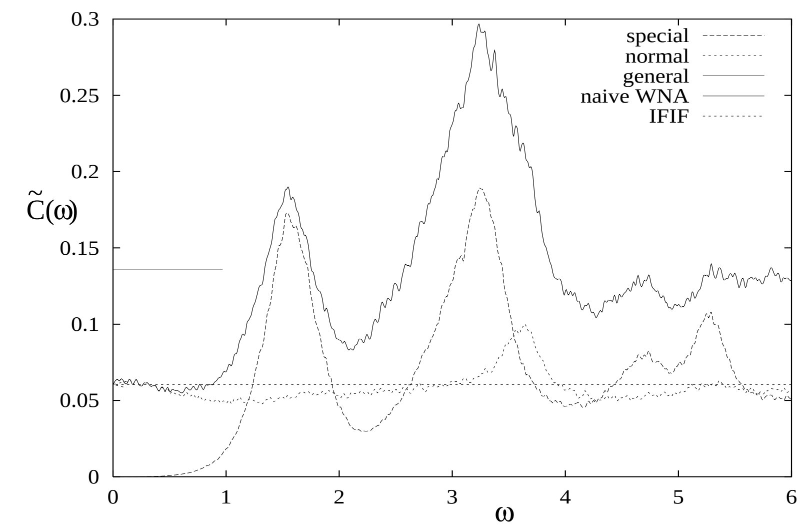
• SUBTRACT SPECIAL COMPONENT:

Make orthonormal set $\{D_{\mathrm{i}}(\mathbf{s})\}$ of special defs, $i = 1 \cdots 1 + \frac{1}{2}d(d+1)$

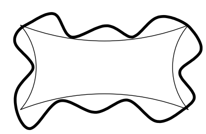
Special projections given by $\alpha_{\mathrm{i}} = \oint d\mathbf{s} D_{\mathrm{i}}(\mathbf{s}) D(\mathbf{s})$

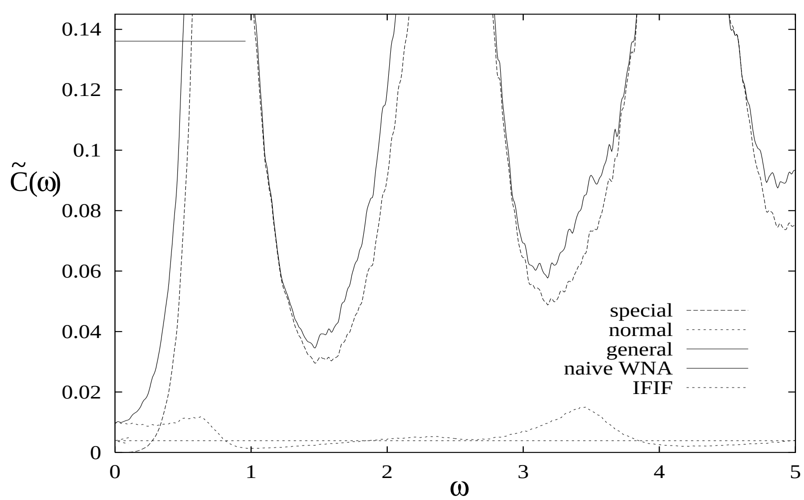
Subtract: $\quad D_{\mathrm{normal}}(\mathbf{s}) \;=\; D(\mathbf{s}) - \sum_{\mathrm{i}} \alpha_{\mathrm{i}}\, D_{\mathrm{i}}(\mathbf{s})$

• NOW APPLY WHITE NOISE ESTIMATE TO $D_{\mathrm{normal}}(\mathbf{s})$:

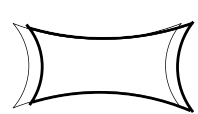


'wave + const'





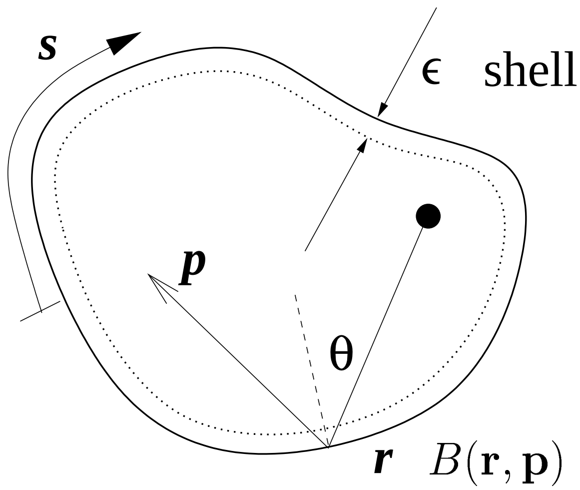
'shift-x'



Conclusions

1. Classical & quantum dissipation rates computed in 2D billiards
   - first study of frequency-dependence in billiards
   - semiclassical correspondence found
   - Applications to driven quantum dots, nuclei…
2. Class of 'special' deformations: power spectrum $\tilde{C}_{\mathrm{E}}(0) = 0$
   - friction coefficient μ vanishes at dc
   - predicts heating rate power laws $\tilde{C}_{\mathrm{E}}(\omega) \sim \omega^{\gamma}$ (*new*)
   - dilation (*new*) $\rightarrow$ eigenstates quasi-orthogonal on boundary (*semiclassical* reason for Vergini numerical method success)
3. Systematic subtraction of 'special' components of general $D(\mathbf{s})$
   - improved upon 25-year-old heating rate estimate
   - subtraction of dilation (*new*)
   - subtraction of trans. & rot. corrects nuclear bad habits

Ergodicity: time average $\to$ boundary integral



$$B(\mathbf{r},\mathbf{p}) \;\equiv\; \begin{cases} |\cos\theta|^3 \; D(\mathbf{s})^2 & \text{within shell} \\ 0 & \text{otherwise,} \end{cases}$$

TIME AVERAGE

$$\text{impulse} \left\{ \begin{array}{rcl} \text{time} & = & t_i \\ \text{value} & = & |\cos\theta_i|^3 \; D(\mathbf{s}_i)^2 \\ \text{duration} & = & 2\epsilon / v \cos\theta_i \end{array} \right.$$

$$\langle B \rangle_t \;=\; \frac{2\epsilon}{v} \left\langle \sum_i \cos^2\theta_i \; D(\mathbf{s}_i)^2 \; \delta(t - t_i) \right\rangle_t$$

PHASE SPACE AVERAGE

$$\text{fraction of position space in shell} \;=\; \frac{\epsilon\,\text{Area}}{\text{Volume}}$$

$$\langle B \rangle_{\mathrm{E}} \;=\; \frac{\epsilon\,\text{Area}}{\text{Volume}} \left\langle |\cos\theta|^3 D(\mathbf{s})^2 \right\rangle_{\mathbf{s},\mathbf{p}}$$

Ergodicity: equate, $\epsilon \to 0$ **:**

$$\left\langle \sum_i \cos^2\theta_i \; D(\mathbf{s}_i)^2 \; \delta(t - t_i) \right\rangle_t \;=\; \frac{v\langle|\cos\theta|^3\rangle}{2\,\text{Volume}} \oint D(\mathbf{s})^2 \, d\mathbf{s}$$

Spreading and irreversible energy growth

Fokker-Planck diffusion for PDF in Ω-space $\rightarrow$ E-space:

$$\dot{\eta} = (D_\Omega \eta')' \qquad \xrightarrow{\text{d.o.s. } g(E)} \qquad \dot{\rho} = \left(g D_\text{E} \left(\frac{\rho}{g}\right)'\right)'$$

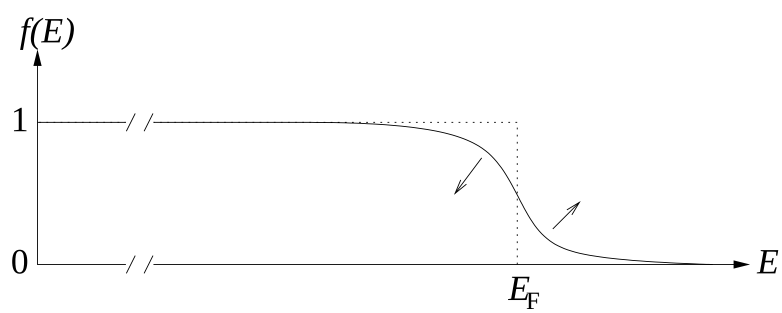
Growth in mean energy:

$$\begin{aligned} \langle \dot{E} \rangle &\equiv \int_0^\infty dE\, E \dot{\rho}(E) \\ &= -\int_0^\infty dE\, g D_\text{E} \left(\left(\frac{\rho}{g}\right)'\right) \\ &= \int_0^\infty dE\, \frac{\rho}{g} (g D_\text{E})' \end{aligned}$$

Gives friction coefficient (general $\rho(E)$)

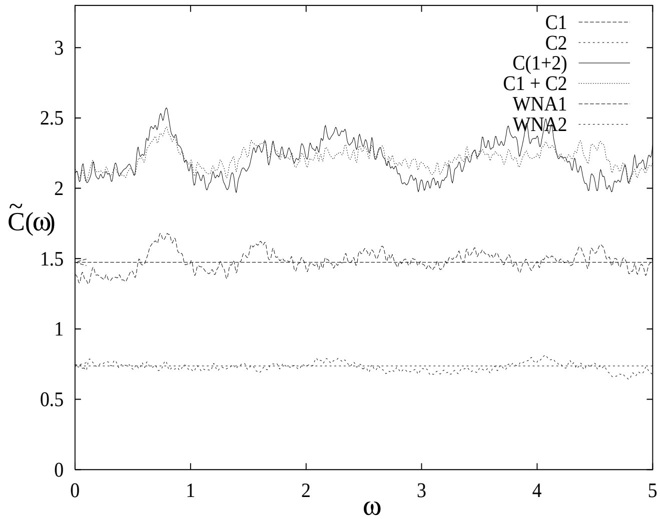
$$\mu(\omega) = \int_0^\infty dE\, \frac{\rho}{g} (g \tilde{C}_\text{E}(\omega))'$$

- Fermi distribution case $\qquad \mu(\omega) = \dfrac{g(E_\text{F}) \tilde{C}_\text{F}(\omega)}{2\Omega_\text{F}}$



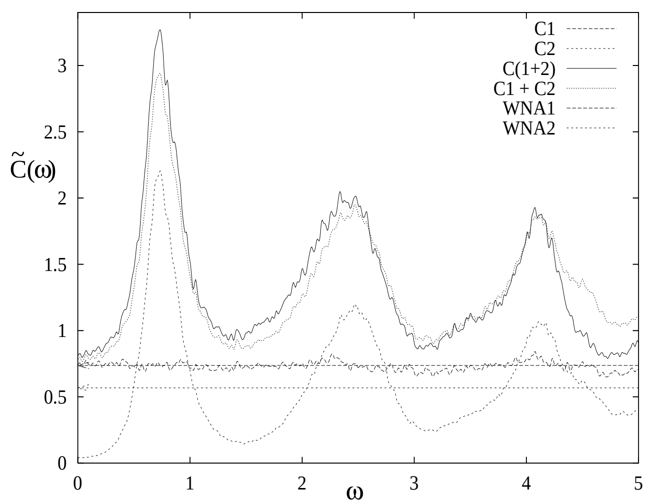
‘Special’ and ‘normal’ details

- Normal (1) + normal (2) = normal (no correlations between $\mathcal{F}$ spikes):



power spectrum of sum is given by WNA

- General (1) + normal (2) (when orthogonal):



$1 \perp 2$

power spectra add

Quality is only as good as ‘goodness’ of 2.

- Special (1) + general (2): $\tilde{C}_{1,2}(\omega \to 0) = \nu_{1,2} = 0$

Follows from special $D(\mathbf{s})$ = zero-eigenvector of ν quadratic form.

Contours of const ν form tubes parallel to *special* directions.

Correspondence and band profile