Quasi-orthogonality on the Boundary, and Deformation of Chaotic Billiards

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Outline. . .

- What is boundary quasi-orthogonality? Why study it?
- Relation to change in hamiltonian when deform billiard
- Semiclassical estimate: classical trajectory correlations
- Why is the band profile a power law?

Quasi-orthogonality

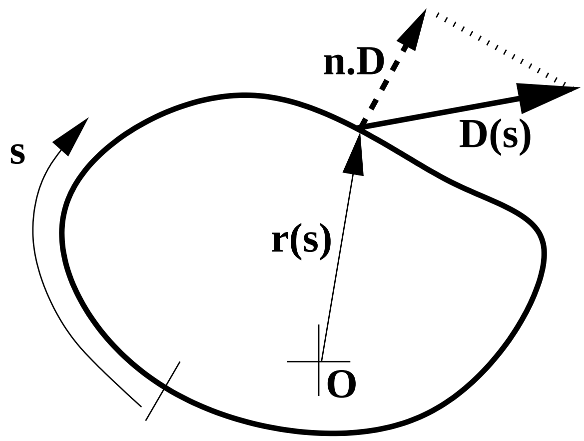
Boundary methods reduce problem from d to $d-1$ dimensions. They are everywhere...

Classical: Poincare Section ($\to$maps)
Quantum: Boundary Integral Method ($\to$kernel)

Normal derivative function on the boundary $\varphi(\mathbf{s}) \equiv \mathbf{n}\cdot\nabla\psi$: contains all information about ψ (*Green's Theorem.*)

φ's are not orthogonal over boundary (*not enough d.o.f.'s*)
Weight the inner product

$$M_{nm} \;\equiv\; \frac{1}{2k^2}\oint \varphi_n(\mathbf{s})\varphi_m(\mathbf{s})\;(\mathbf{n}\cdot\hat{\mathbf{D}})d\mathbf{s} \tag{1}$$



Special case $\mathbf{D}(\mathbf{s}) = \mathbf{r}(\mathbf{s})$: $M_{nm} \approx \delta_{nm}$ *"quasi-orthogonality"*

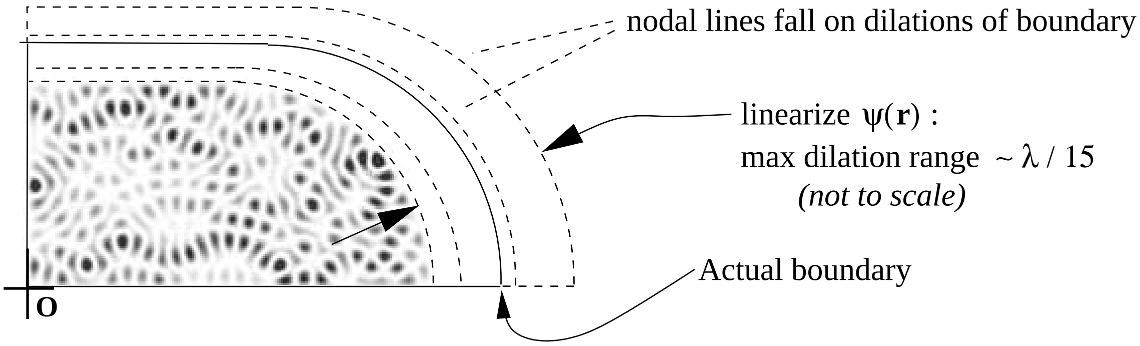
($m = n$ Berry-Wilkinson, $m \neq n$ Vergini-Saraceno)

We'll see $\mathbf{D}(\mathbf{s})$ can be thought of as a "deformation field" which moves the billiard wall...

Motivation: Why care about M_{nm} ?

1. VERGINI METHOD

If had 'complete' computational basis $\{\phi\}$ for Helmholtz solutions at a single $k = k_0$:
Could represent all true eigenstates $\{\psi_\mu(k_\mu)\}$ at once by *rescaling* them to the same wavenumber $k_\mu \to k_0$.

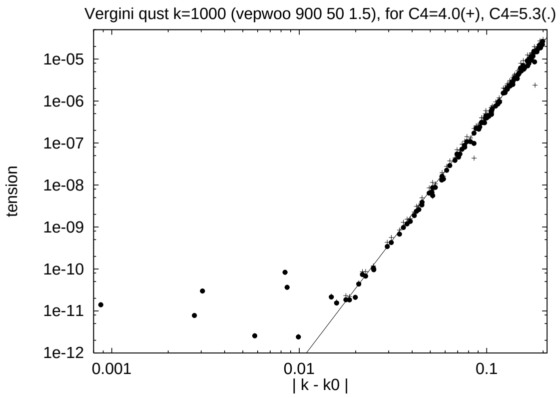


Specially-weighted *tension* measured on actual boundary:

$$f(k_0) \equiv \oint \psi^2 d\mathbf{s}/r_n \quad \underset{\text{basis}}{\overset{\{\psi_\mu(k_0)\}}{\longrightarrow}} \quad 2(k_0 - k_\mu)(k_0 - k_\nu) M_{nm} \approx \text{diagonal}$$

METHOD: Diagonalize matrices $f(k_0)$, df/dk_0 in $\{\phi\}$ basis:

gives transformation $\{\phi\} \to \{\psi_\mu(k_0)\}$, in limited range $|k_\mu - k_0| < r_{max}/5$

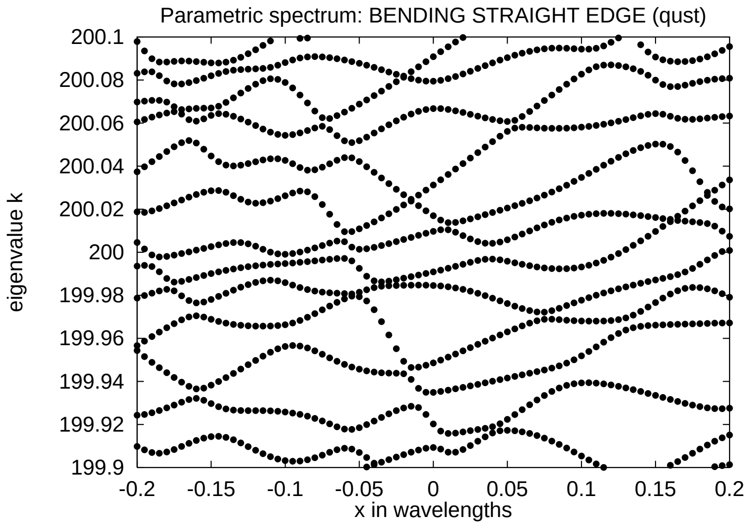


How explain tension errors $\sim |k - k_0|^6$?

2. DRIVEN SYSTEM DISSIPATION

Average $|M_{nm}|^2 \to$ rate of irreversible energy absorption when boundary deformed at freq $\omega = (E_n - E_m)/\hbar$ (Kubo-Greenwood)

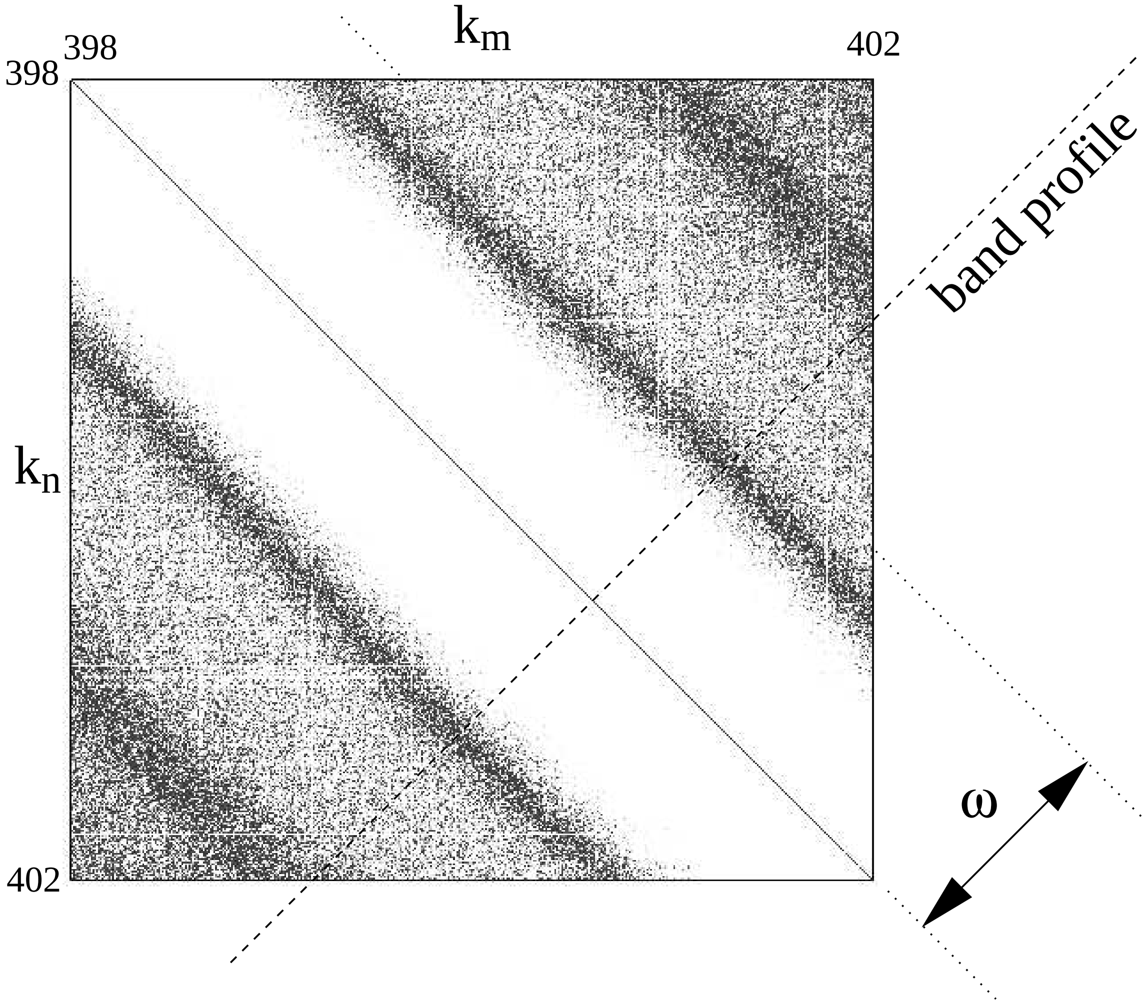
3. DIABOLICAL POINTS



Parametric spectrum:
$\delta x_c^{qm} = \frac{\Delta}{\sigma}$
with $\sigma =$ rms $M(\omega \to 0)$

$$\text{density of} \begin{cases} \text{avoided crossings} \sim \sigma \\ \text{diabolical points} \sim \sigma^2 \text{ (Wilkinson-Austin, GOE)} \end{cases}$$

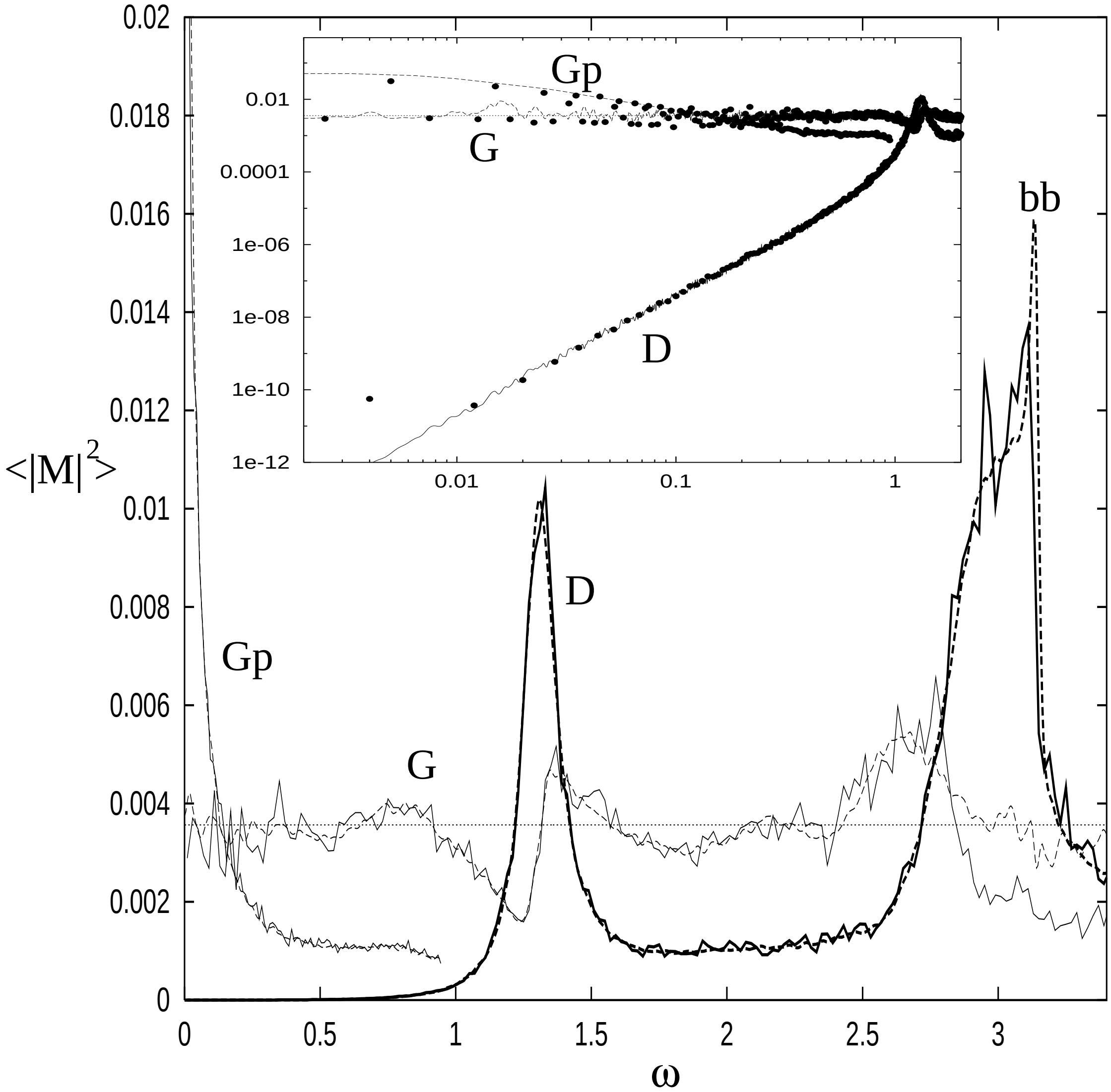
What does M_{nm} look like?



451 states of Quarter Stadium, $\mathbf{D}(\mathbf{r}) = \mathbf{r}$.

Now average $|M(\omega)|^2$ over classically small k range...

Band Profile of $|M_{nm}|^2$:



D — deformation is dilation: $\mathbf{D}(\mathbf{s}) = \mathbf{r}(\mathbf{s})$

G — generic deformation (sinusoidal wobble)

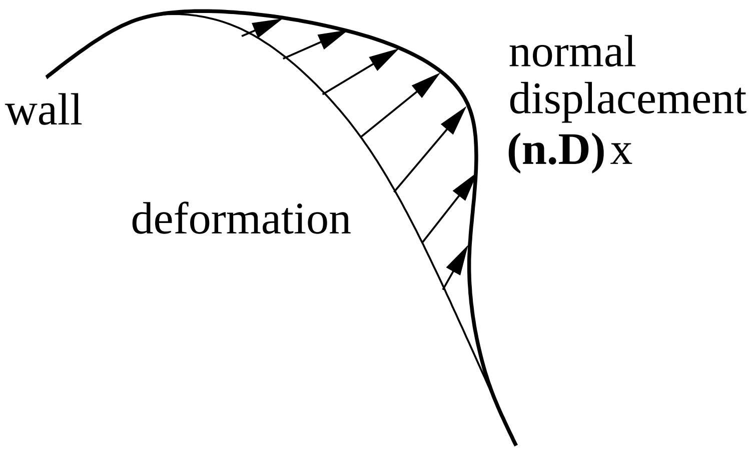
- system-specific structure at $\omega \sim 1/\tau_{col} \sim 1$
- different possible 'universal' $\omega \to 0$ tails

Uncorrelated random wave estimate for off-diagonal M, general $\mathbf{D}(\mathbf{s})$:

$$\frac{M_{nm}}{M_{nn}} \sim \frac{\text{overlap of } \varphi_m \text{ with } \varphi_n}{\text{self-overlap of } \varphi_n} \sim \begin{pmatrix}\text{number of } \lambda_{\mathrm{B}}\text{-sized} \\ \text{patches on boundary}\end{pmatrix}^{1/2} = \text{const}$$

For dilation (& some others), wrong $\omega \to 0$ behaviour... mysterious?

Relate M_{nm} to billiard wall deformation:



Parameter change δx:
change in Hamiltonian $\delta\mathcal{H}$
nonzero only on boundary.
Can be expressed in φ_m

Take limit wall potential $V_0 \to \infty$:

$$M_{nm} = -\frac{m}{(\hbar k)^2}\left(\frac{\partial \mathcal{H}}{\partial x}\right)_{nm} \tag{2}$$

Semiclassical estimate for $(\partial\mathcal{H}/\partial x)_{nm}$:

Classical time-dependent quantity: the "force" on parameter x:

$$\mathcal{F}(t) \;=\; -\frac{\partial\mathcal{H}(\mathbf{r}(t),\mathbf{p}(t);x)}{\partial x} = \sum_{col} 2mv\;\cos(\theta_{col})\; D_{col}\;\delta(t-t_{col})$$

Ergodicity of (almost all) eigenstates (Shnirlman): for any O,

$$\begin{aligned}\mathrm{Tr}[\hat{O}] \equiv \frac{1}{N}\sum_{n=1}^{N}\langle n|\hat{O}|n\rangle \;&\approx\; \langle O(P,Q)\rangle_E \text{ microcanonical average}\\ &=\; \langle O\rangle_t \text{ time average,ergodic}\end{aligned}$$

Choose $\hat{O} = \hat{\mathcal{F}}(t)\hat{\mathcal{F}}(t+\tau)$, gives correlation function:

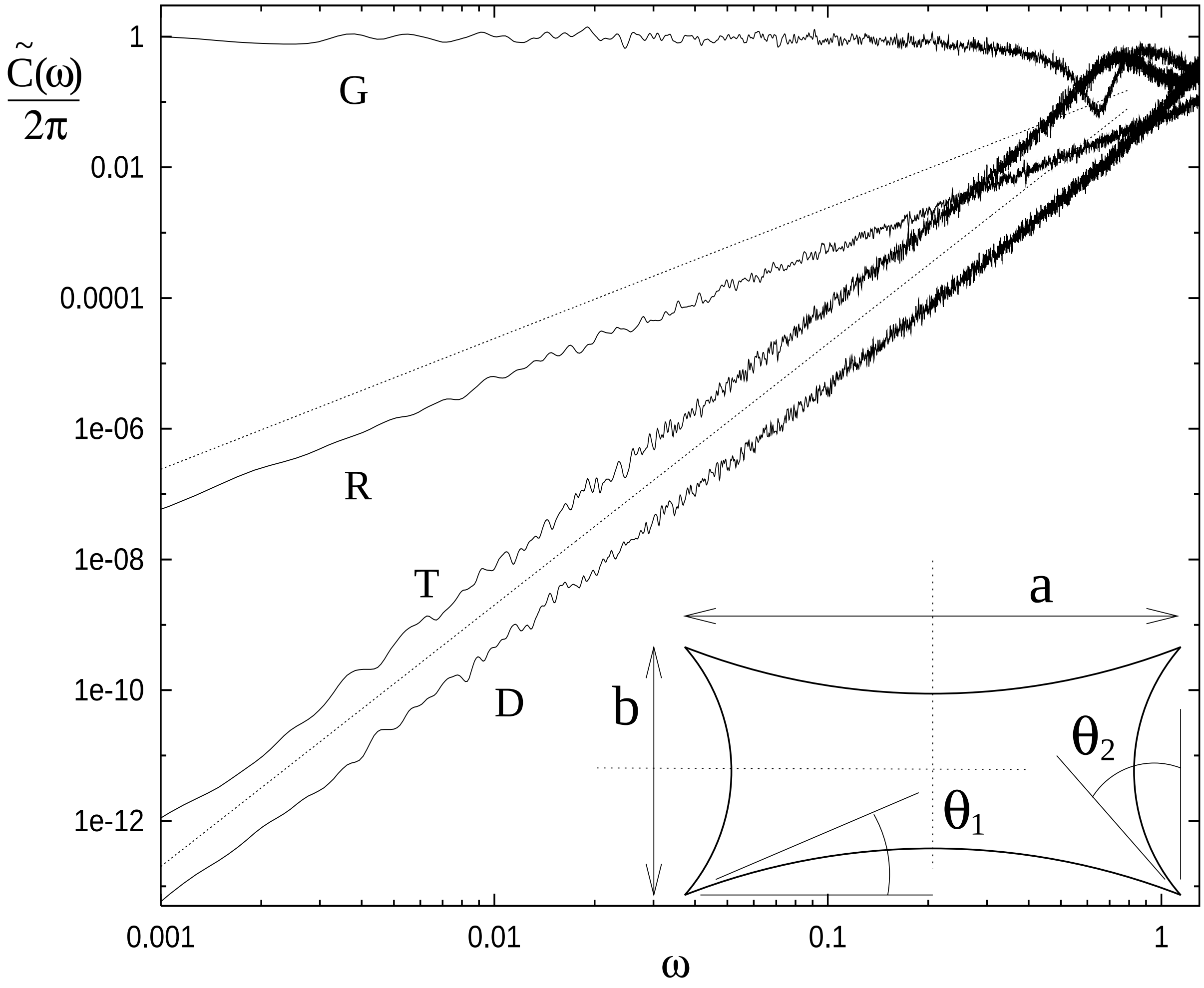
$$\begin{aligned}C(\tau) \;&\equiv\; \langle\mathcal{F}(t)\mathcal{F}(t+\tau)\rangle_t\\ &\approx\; \frac{1}{N}\sum_{nm}\langle n|\hat{\mathcal{F}}(t)|m\rangle\langle m|\hat{U}(\tau)\hat{\mathcal{F}}(t)\hat{U}^{\dagger}(\tau)|n\rangle\\ &=\; \sum_{m}\frac{1}{N}\sum_{n=1}^{N}|\mathcal{F}_{nm}|^2 e^{i(E_n-E_m)\tau/\hbar}\end{aligned}$$

Take Fourier transform, average over n:

$$\left\langle\left|\left(\frac{\partial\mathcal{H}}{\partial x}\right)_{nm}\right|^2\right\rangle \;\equiv\; \langle|\mathcal{F}_{nm}|^2\rangle \;\approx\; \frac{\Delta}{2\pi\hbar}\,\tilde{C}\left(\frac{E_n-E_m}{\hbar}\right)$$

Estimate $\tilde{C}(\omega)$ with long classical billiard trajectory...

Classical $\tilde{C}(\omega)$ in Generic Billiard



G — generic deformation ω^0
R — rotation ω^2
D — dilation
T — translation

$\left.\begin{matrix}\text{D} \\ \text{T}\end{matrix}\right\} \quad \omega^4$

Explain Power Laws for $\tilde{C}(\omega)$

The generalized "force" $\mathcal{F}(t) = -\mathbf{D}(\mathbf{r}) \cdot \dot{\mathbf{p}}$.

$\mathbf{D}(\mathbf{r})$ = deformation field (over all space),
$\dot{\mathbf{p}}$ = rate of change of particle momentum.

Special cases, can show $\mathcal{F}(t)$ is an exact derivative of something...

TRANSLATION: $\mathbf{D} = \vec{\mathbf{e}}$, const

can write $\mathcal{F}(t) = (d/dt)^2 \mathcal{G}(t)$, with $\mathcal{G}(t) = -m\vec{\mathbf{e}} \cdot \mathbf{r}$.

DILATION: $\mathbf{D} = \mathbf{r}$

same with $\mathcal{G}(t) = -(m/2)\mathbf{r}^2$

ROTATION: $\mathbf{D} = \vec{\mathbf{e}} \times \mathbf{r}$

can write $\mathcal{F}(t) = (d/dt)\mathcal{G}(t)$, with $\mathcal{G}(t) = -\vec{\mathbf{e}} \cdot (\mathbf{r} \times \mathbf{p})$.

Assume $\mathcal{G}(t)$ has white noise spectrum (ω^0), gives observed power laws.

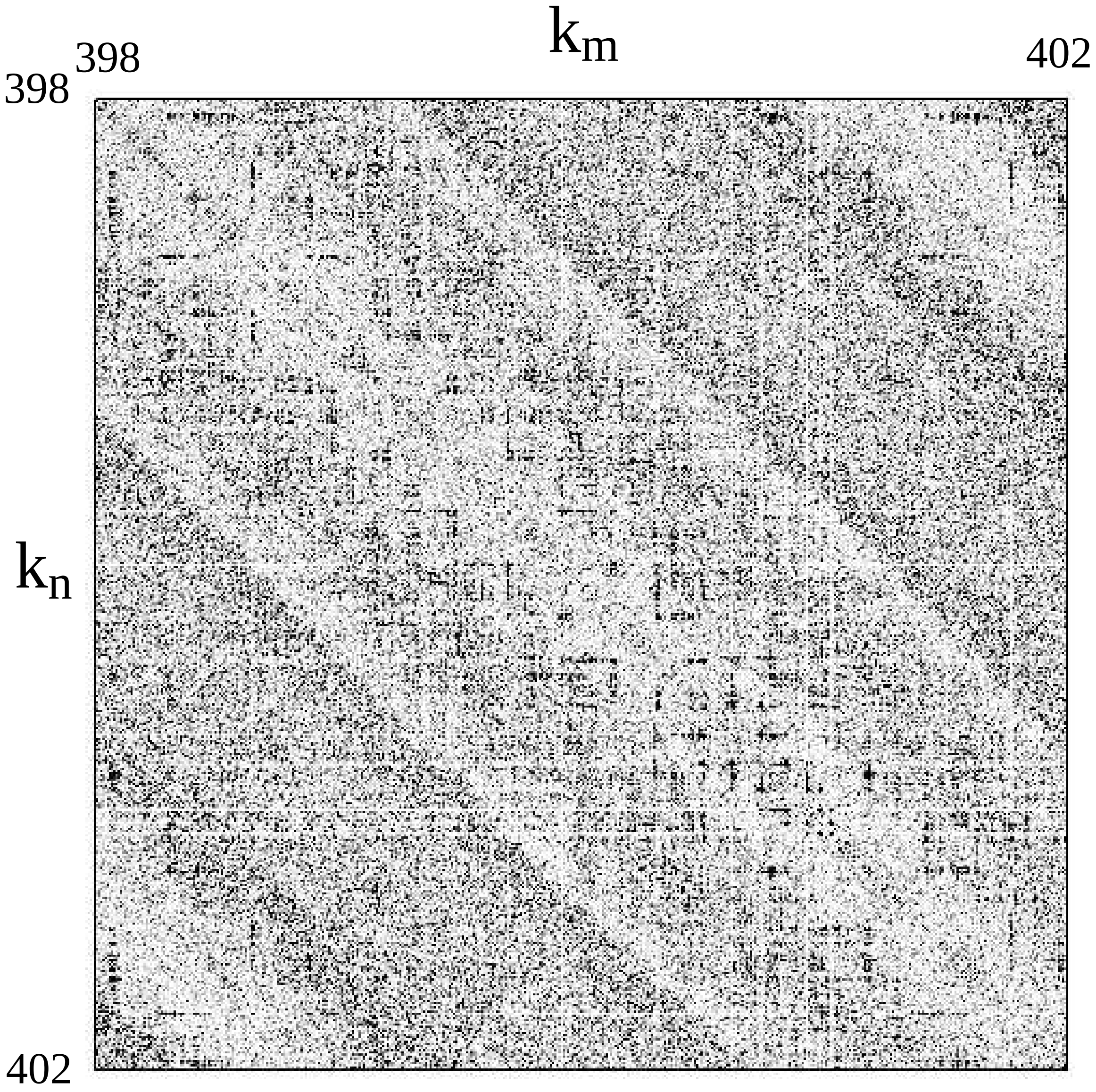
Complications: non-generic billiards ("bouncing ball" in stadium) $\rightarrow$ long-time correlations in $\mathcal{G}(t)$.

Conclusion

- Basis for quasi-orthogonality of boundary eigenstates in chaotic classical dynamics $\rightarrow$ billiard properties.
- Important numerical consequences for Vergini method
- Found 3 special time-dependent deformations with vanishing dissipation coefficient as $\omega \rightarrow 0$. (Note dissipation *rate* $\sim$ coefficient $\cdot\omega^2$.)

Future Questions

- Non-gaussian distribution of M_{nm} elements:



- Relate prefactors of power laws to physical billiard properties.
- Predicting $\tilde{C}(\omega \to 0)$ for a given deformation, "subtracting out" D, T, R components.
- Accuracy of Vergini Method given deviation $M_{nm} - \delta_{nm}$.