Robust and efficient computation of two-dimensional photonic crystal band structure using second-kind integral equations

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Photonic crystals

periodic dielectric structures
period ≈ wavelength of light ≈ 1μm
control optical propagation in ways impossible in homogeneous media

(Joannopoulos group)
Photonic crystals

periodic dielectric structures
period \( \approx \) wavelength of light \( \approx 1\mu m \)
control optical propagation in ways impossible in homogeneous media

e.g. ‘bandgap’ medium: \( \exists \) freqs. s.t.
all waves evanescent (non-propagating)
- ‘insulators’ with embedded waveguides
- unlike dielectric guides, sharp bends ok

2D lattice of cylinders (INFM, U. Pavia)
Photonic crystal examples

- Slab w/ 2D-periodic air holes couples to external dielectric guide
  manipulate guide dispersion to give very slow group velocity \((c/300)\)

Si, \(\lambda = 1.6\mu m\) (Vlasov ’05)
Photonic crystal examples

- Slab w/ 2D-periodic air holes couples to external dielectric guide manipulate guide dispersion to give v slow group velocity ($c/300$)

Si, $\lambda = 1.6\mu m$ (Vlasov ’05)

- Full 3D bandgap (all polarizations)
- ‘Yablonovite’ (cm scale) (Yablonovich ’91)
- ‘woodpile’ $\lambda = 12\mu m$ (Lin et al. ’98)
- ‘inverse opals’ (spherical air ‘bubbles’) stacked slabs (built $\lambda = 1.3\mu m$, Qi et al. ’04)
- complex geometry (not just cylinders!)

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Applications

Build low-loss optical signal paths on 1µm scale: integrated optical devices, signal-processing,
Big goal: optical (high speed!) computing
e.g. high-Q resonators, couplers, junctions
channel-drop filter in 2D crystal

(Johnson et al. ’00)
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- Meta-materials e.g. negative refractive index (−1 = ‘perfect’ lens)
- Solar cells and LEDs: control the density of states
  ⇒ spontaneous emission/absorption rates, directions (S. Fan ’97)
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Common features

• piecewise-homogeneous dielectric media, wavenumber low
• each medium linear, may be dispersive e.g. metals (plasmons)
• manufacturing costly \(\Rightarrow\) accurate numerical modeling key
Outline

1. Band structure: eigenmodes on a torus
2. Boundary integral equations
3. Periodizing: standard way & new way
4. Interpolation of bands
5. Software environment
2D dielectric crystal (z-invariant Maxwell, TM polarization)

unit cell $U$ smooth inclusion $\Omega \subseteq U$, refractive index $n$

lattice $\Lambda := \{me_1 + ne_2 : n, m \in \mathbb{Z}\}$

dielectric inclusions $\Omega_\Lambda := \{x : x + d \in \Omega \text{ for some } d \in \Lambda\}$
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scalar wave $u : \mathbb{R}^2 \to \mathbb{C}$ obeys

PDE (fixed frequency $\omega$):

$$(\Delta + n^2 \omega^2)u = 0 \text{ in } \Omega_\Lambda$$
$$(\Delta + \omega^2)u = 0 \text{ in } \mathbb{R}^2 \setminus \Omega_\Lambda$$

material matching conditions:

$$u^+ - u^- = 0 \text{ on } \partial \Omega_\Lambda$$
$$u_{n}^+ - u_{n}^- = 0 \text{ on } \partial \Omega_\Lambda$$
Bloch ‘theorem’

Solutions to PDE w/ periodic coeffs have form (or are sum of forms)

\[ u(x) = e^{ik \cdot x} \tilde{u}(x), \quad \tilde{u} \text{ is periodic} \]

- called Bloch waves, \( k \in \mathbb{R}^2 \) Bloch wavevector

‘When I started to think about it, I felt that the main problem was to explain how the electrons could sneak by all the ions in a metal... By straight Fourier analysis I found to my delight that the wave differed from the plane wave of free electrons only by a periodic modulation’

(F. Bloch, 1928)

(indep. prediscovered by Hill 1877, Floquet 1883, Lyapunov 1892)
Bloch wave and eigenvalue problem

- Bloch eigenvalues: set of $(\omega, k)$ s.t. non-trivial Bloch waves $u$ exist
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Example generalized eigenfunction \(u\) of Bloch wave form \(e^{ik \cdot x} \tilde{u}(x)\):

Shown: \(\text{Re}[u]\) for
\(\omega = 5, \ k = (-0.39, 2.08)\)
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Shown: Re\([u]\) for \(\omega = 5, \quad \mathbf{k} = (-0.39, 2.08)\)

\(\mathbf{k}\) equiv. to \(\mathbf{k} + \mathbf{q}, \quad \forall \mathbf{q} \in 2\pi \Lambda^*\)

\(\Lambda^* = \text{dual (reciprocal) lattice}\)

\(\mathbf{k}\) lives on a torus, consider only fundamental domain (FD):
Band structure

For each wavevector $\mathbf{k} \in \text{FD}$, there exist discrete Bloch eigenvalues

$$\omega_1(\mathbf{k}) \leq \omega_2(\mathbf{k}) \leq \cdots \to \infty$$

- form ‘sheets’ above the FD

note: conical at low freq $\omega$

note: bandgap

- is most important property of photonic crystal for applications
Recast problem on compact domain (torus)

- Bloch wave condition equiv. to quasi-periodic BCs on $\partial U$

Require vanishing unit cell discrepancy:

\[
\begin{align*}
    f & := u|_{L} - \alpha^{-1}u|_{L+e_1} = 0 \\
    f' & := u_n|_{L} - \alpha^{-1}u_n|_{L+e_1} = 0 \\
    g & := u|_{B} - \beta^{-1}u|_{B+e_2} = 0 \\
    g' & := u_n|_{B} - \beta^{-1}u_n|_{B+e_2} = 0
\end{align*}
\]

Bloch phase parameters $\alpha := e^{i k \cdot e_1}$, $\beta := e^{i k \cdot e_2}$, $|\alpha| = |\beta| = 1$

- Task: find Bloch eigenvalue triples $(\omega, k_x, k_y)$, i.e. $(\omega, \alpha, \beta)$
Main numerical approaches

Time domain

a) time-stepping on finite-difference grid (FDTD)  
   low order (inaccurate); close freqs → need large $t$ (inefficient)  

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Freq domain
b) multiple-scattering, KKR, cylinders only \( (McPhedran, \ Moroz) \)
c) Plane-wave method: all in Fourier space \( (Joannopoulos, \ Johnson, \ Sözüer) \)
   discont. dielectric ⇒ Gibbs phenom, slow \( (1/N \ or \ 1/N^2) \) convergence

d) Finite element (FEM) discretization in \( U \) \( (Chew, \ Dobson, \ Dossou) \)
   better for discontinuity, \( N \) large, meshing complicated
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e) Integral equations: formulate problem \(on\) the discontinuity \(\partial \Omega\)
   reduced dimensionality (small \(N\))
   high order (quadratures): high accuracy \(w/\) small effort \((\Rightarrow\) sensitivity analysis\) \((Yuan ~'08)\)
Potential theory

‘charge’ (sources of waves) distributed along curve $\Gamma$ w/ density func.

single-, double-layer potentials, $x \in \mathbb{R}^2$:

$$u(x) = \int_{\Gamma} \Phi_\omega(x, y) \sigma(y) dy := (S\sigma)(x)$$

$$v(x) = \int_{\Gamma} \frac{\partial \Phi_\omega}{\partial n_y}(x, y) \tau(y) dy := (D\tau)(x)$$

$$\Phi_\omega(x, y) := \Phi_\omega(x - y) := \frac{i}{4} H_0^{(1)}(k|x - y|)$$

Helmholtz fundamental soln aka free space Greens func
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Helmholtz fundamental soln aka free space Greens func

Jump relations: limit as $x \to \Gamma$ may depend on which side ($\pm$):

\[ u_{\pm} = S\sigma \]
\[ u_{\pm}^n = D^T\sigma \mp \frac{1}{2}\sigma \]
\[ v_{\pm} = D\tau \pm \frac{1}{2}\tau \]
\[ v_{\pm}^n = T\tau \]

$S, D$ are integral ops with above kernels but defined on $C(\Gamma) \to C(\Gamma)$

$T$ has kernel $\frac{\partial^2 \Phi_\omega}{\partial n_x \partial n_y}(x, y)$
Integral equations for scattering (sketch)

e.g. Dirichlet obstacle: represent $u = u^{\text{inc}} + \mathcal{D} \tau$

DLP on $\partial \Omega$
Integral equations for scattering (sketch)

e.g. Dirichlet obstacle: represent \( u = u^{\text{inc}} + D\tau \)

\[ \text{BC: } 0 = u^+ = u^{\text{inc}}|_{\partial \Omega} + (D + \frac{1}{2})\tau \]

integral eqn on \( \partial \Omega \): \((I + 2D)\tau = -2u^{\text{inc}}\)

2nd-kind, \( D \) compact op so \((I + 2D)\) sing. vals. \( \not\to 0 \)

Why important? when scale up...

condition # small, iterative solvers (GMRES) fast
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Why important? when scale up... condition # small, iterative solvers (GMRES) fast

Quadrature scheme: choose \( N \) nodes \( y_j \in \partial \Omega \), weights \( w_j \)

Nyström discretization: \( N \)-by-\( N \) linear system for vector \( \{\tau^{(N)}_k\}_{k=1}^N \)
\[
\tau^{(N)}_k + 2\sum_{j=1}^N w_j D(y_k, y_j)\tau^{(N)}_j = -2u^{\text{inc}}(y_k), \quad k = 1, \ldots, N
\]
Integral equations for scattering (sketch)

**e.g.** Dirichlet obstacle: represent \( u = u^{\text{inc}} + D\tau \) 

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**Thm:** (Anselone, Kress) \( \|\tau^{(N)} - \tau\|_{\infty} \) converges at same rate as quadrature scheme for the true integrand \( D(y, \cdot)\tau \).

- **Analytic curve & data, periodic trapezoid rule:** spectral convergence
- **e.g.** above: \( N = 60 \) enough for \( 10^{-6} \) error, \( N = 100 \) for \( 10^{-12} \)
- **error** = \( O(e^{-\gamma N}) \), rate \( \gamma \approx \) distance to nearest singularity of \( \tau \) in \( \mathbb{C} \)
Dielectric (transmission) scattering

Represent $u = u^{\text{inc}} + D\tau + S\sigma$ outside wavenumber $\omega$

$u = D_i\tau + S_i\sigma$ inside wavenumber $n\omega$
Dielectric (transmission) scattering

Represent $u = u^{\text{inc}} + D\tau + S\sigma$ outside wavenumber $\omega$
$u = D_i\tau + S_i\sigma$ inside wavenumber $n\omega$

mismatch on $\partial\Omega$: $h := u^+ - u^-$, $h' := u^+_n - u^-_n$

BCs: mismatch $m := [h; h']$ vanishes, use JRs...

$$
\begin{bmatrix}
0 \\
0
\end{bmatrix} = 
\begin{bmatrix}
u^{\text{inc}}|_{\partial\Omega} \\
u^{\text{inc}}_n|_{\partial\Omega}
\end{bmatrix}
+ 
\begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix}
+ 
\begin{pmatrix}
D - D_i & S_i - S \\
T - T_i & D_i^T - D^T
\end{pmatrix}
\begin{bmatrix}
\tau \\
-\sigma
\end{bmatrix}
$$

block 2nd-kind

$A$ maps densities to their effect on mismatch

- hypersingular part of $T$ cancels, so $A = \text{Id} + \text{compact}$ (Rokhlin '83)
- kernel weakly singular, but exists spectral product quadrature
  for $f(s) + \log(4\sin^2\frac{s}{2})g(s)$, $f, g$ analytic $2\pi$-periodic (Kress '91)
The standard way to periodize

replace kernel $\Phi_\omega(x)$ by $\Phi_{\omega,\text{QP}}(x) := \sum_{m,n \in \mathbb{Z}} \alpha^m \beta^n \Phi(x - me_1 - ne_2)$

thus integral operator $A$ becomes $A_{\text{QP}}$
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- is standard approach for scattering from gratings (McPhedran, Otani)
- seems natural for band structure problem . . . not yet used in literature
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**Theorem** (*integral formulation of band structure*):

\[
\text{If } A_{\text{QP}} \text{ exists, } \text{Nul } A_{\text{QP}} \neq \{0\} \iff (\omega, k_x, k_y) \text{ is eigenvalue}
\]

note: no $u^{\text{inc}}$ since eigenvalue problem homogeneous
proof: non-physical fields are a swapped-media transmission BVP
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proof: non-physical fields are a swapped-media transmission BVP

Not a robust method: $A_{\text{QP}}$ does not exist for certain parameters $(\omega, k_x, k_y)$ since there $\Phi_{\omega,\text{QP}}(x) \to \infty, \forall x$

why...?
Failure at spurious resonances

\( \Phi_{\omega,QP}(x) \) is Helmholtz Greens function in empty (index 1) torus

\[
\frac{1}{\text{Vol}(U)} \sum_{\mathbf{q} \in 2\pi\Lambda^*} \frac{e^{i(k+q) \cdot x}}{\omega^2 - |k + q|^2}
\]

spectral representation on torus

has simple pole wherever \((\omega, k_x, k_y)\) is eigenvalue of empty torus... but physical field \(u\) well-behaved here: breakdown is non-physical!
Failure at spurious resonances

\( \Phi_{\omega, QP}(x) \) is Helmholtz Greens function in *empty* (index 1) torus

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\frac{1}{\text{Vol}(U)} \sum_{q \in 2\pi \Lambda^*} \frac{e^{i(k+q) \cdot x}}{\omega^2 - |k + q|^2}
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Our cure: robust way to periodize

\[ u = D\tau + S\sigma + \text{(densities } \xi \text{ on walls of } U) \text{ outside} \]

\[ \uparrow \quad \uparrow \]

\[ \text{can enforce mismatch } m = 0 \quad \text{can enforce discrepancy } d := [f; f'; g; g'] = 0 \]
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represent \( u = D\tau + S\sigma + (\text{densities } \xi \text{ on walls of } U) \) outside

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can enforce mismatch \( m = 0 \) \quad can enforce discrepancy \( d := [f; f'; g; g'] = 0 \)

In block operator form

\[
\begin{bmatrix}
A & B \\
C & Q
\end{bmatrix}
\begin{bmatrix}
\eta \\
\xi
\end{bmatrix}
= 
\begin{bmatrix}
m \\
d
\end{bmatrix}
\]

- added extra degrees of freedom (a small #, indep. of complexity of \( \Omega \))
Our cure: robust way to periodize

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- added extra degrees of freedom (a small #, indep. of complexity of $\Omega$)
- gain robustness: no matrix element blow-up at spurious resonances

Observe: $\text{Nul } M \neq \{0\} \iff (\omega, k_x, k_y) \text{ Bloch eigenvalue}$

- idea of extra sources of waves not new (e.g. Hafner ’02)
- what is new: $M = \text{Id} + \text{compact}$ ideal for large-scale, iterative, FMM
How choose new densities on unit cell walls?

- to control 4 discrepancies $(f, f', g, g')$
- need 4 densities $\xi = [\tau_L; \sigma_L; \tau_B; \sigma_B]$

$Q = \frac{1}{2} \text{Id} + \text{(self-interactions)} + \text{(other interactions)}$

\begin{align*}
\text{JRs} & \quad \sigma_L \rightarrow u|_L \\
\tau_L, \sigma_L & \quad \tau_L, \sigma_B \\
\tau_B, \sigma_B &
\end{align*}
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  \(\text{JRs} \quad \sigma_L \rightarrow u|_L \quad \sigma_L \rightarrow u|_B\)

- add phased ghost copies on other 2 walls
  recall \(f := u|_L - \alpha^{-1}u|_{L+e_1}\)
  effect of \(\sigma_L\) on \(u_n|_L\)
  effect of \(\alpha \sigma_L\) on \(\alpha^{-1}u_n|_{L+e_1}\) \(\text{cancel}\) apart from Id
How choose new densities on unit cell walls?

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  effect of \(\alpha \sigma_L\) on \(\alpha^{-1} u_n|_{L+e_1}\) } cancel apart from \(\text{Id}\)

- add more ‘sticking-out’ ghost images

  effect of \(\neq\) on \(u_n|_L\)
  effect of \(\neq\) on \(\alpha^{-1} u_n|_{L+e_1}\) } cancel apart from \(\text{Id}\)

  \(\Rightarrow\) all corner interactions vanish!

- result: \(Q = I + (\text{interactions of distance } \geq 1)\)

  \(\Rightarrow\) low rank, rapid convergence: 20-pt Gauss quadr. on \(L, B \Rightarrow 10^{-12}\) error
Full scheme
Finally we add 3x3 phased image copies of densities on $\partial\Omega$, giving:
Full scheme

Finally we add 3x3 phased image copies of densities on $\partial \Omega$, giving:

- Careful cancellations: $B, C, Q$ have only interactions of distance $\geq 1$
- Large dist increases convergence rate, i.e. large $c$ in error = $O(e^{-cN})$

*Philosophy:* sum neighboring image sources directly so fields due to remainder of lattice have distant singularities
Error convergence

$log_{10} \min \text{ sing. val } M$ for known Bloch eigenvalue (should be zero):

Note: is eigenvalue error up to $O(1)$ const

$\omega = 5, \mathbf{k} \approx (-0.39, 2.08)$

- Spectral (exponential) convergence in inclusion & wall # dofs
Crude results: small inclusion

band structure: simply plot log min sing. val. of $M$ vs $(\omega, k_x, k_y)$ ...
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band structure: simply plot log min sing. val. of $M$ vs $(\omega, k_x, k_y)$ ...

- 0.1 sec per eval
- pre-store $\alpha, \beta$ coeffs
- 30 sec per const-$\omega$ slice
- $24 \times 24$ evals

- errors $10^{-9}$ for 40 pts on $\partial \Omega$, 20 on each wall (total $N = 160$)
Large inclusion passing through unit cell

As \( \text{dist}(\Omega, \partial U) \rightarrow 0 \) standard quadrature v. poor
- fix via adaptive quadrature of Lagrange interpolant
- faster: project wall densities onto J-expansion using Graf addition thm (needs \( N = 35 \) per wall)
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Amazingly (due to far singularities), J-exp analytically continues the field to outside \( U \):

\[
\omega = 4.47
\]

\[
k \approx (0.17, 2.11)
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\( n=1 \) inside

\( n=3.3 \) outside
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$$n=1 \text{ inside}$$
$$n=3.3 \text{ outside}$$

Sampling fine 3D grid is crude & slow: how find bands to spectral acc?
Interpolation across the Brillouin zone

How find eigenvalue sheets in the volume $S^1 \times S^1 \times (0, \omega_{\text{max}})$?
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- slow (20 its per root), unreliable (misses nearby roots)
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Realize: $M = I + \text{(cpt op-valued analytic func of } \omega, k_x \text{ and } k_y)$

$\det M$ is a Fredholm determinant, also analytic

- rootfinding a real-analytic function is nice... (J. Boyd ’02)
Spectral rootfinding of analytic functions

\( f : \mathbb{R} \to \mathbb{R}, \ 2\pi\text{-periodic} \)
Spectral rootfinding of analytic functions

\( f : \mathbb{R} \rightarrow \mathbb{R}, \quad 2\pi\text{-periodic} \)

use trigonometric poly interpolant

\[
f(\theta) \approx \sum_{n=-N}^{N} c_n e^{in\theta}
\]

exponentially convergent in \( N \)
width of strip about \( \mathbb{R} \) in which \( f \) analytic
Spectral rootfinding of analytic functions

\( f : \mathbb{R} \rightarrow \mathbb{R}, \ 2\pi\text{-periodic} \)

use trigonometric poly interpolant

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- \( c_n \) found by FFT of grid samples \( f(\pi n / N), \ n = 1, \ldots, 2N \)
Spectral rootfinding of analytic functions

(Boyd ’02)

$f : \mathbb{R} \rightarrow \mathbb{R}$, $2\pi$-periodic

use trigonometric poly interpolant

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width of strip about $\mathbb{R}$ in which $f$ analytic

- $c_n$ found by FFT of grid samples $f(\pi n/N), \ n = 1, \ldots, 2N$

map $e^{i\theta} = z \in \mathbb{C}$

$\Rightarrow$ Laurent $q(z) = \sum_{n=-N}^{N} c_n z^n$

roots of $f$ lie on $|z| = 1$

has roots near $|z| = 1$
Spectral rootfinding of analytic functions

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has roots near \( |z| = 1 \)

“Degree doubling”: \( z^N q(z) \) is degree-\( 2N \) poly, so…

\( \bullet \) use Matlab \texttt{roots\ QR} for eigvals of companion matrix, \( O(N^3) \) but v. stable

\( \bullet \) extract the angles \( \theta \) of roots near unit circle

(Boyd nonlin EVP; Trefethen-Battles ’06 \texttt{chebfun})
Rootfinding det $M$ in the $\omega$ direction

$\det M(\omega, k_x, k_y)$ not periodic in $\omega$: map $\omega = \omega_0 + a \cos \theta$ periodic $\theta$

this is Chebyshev interpolation on interval $[\omega_0 - a, \omega_0 + a]$

Fix $k$, eval $\det M$ at Cheby pts, get $\omega_j(k)$ in interval

25 evals covers $\omega \in [4, 6]$, i.e. 10-20 evals per root found
Rootfinding \( \det M \) in the \( \omega \) direction

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Fix \( k \), eval \( \det M \) at Cheby pts, get \( \omega_j(k) \) in interval

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Also analytic in \( k_x, k_y \Rightarrow \) interpolate in 3D!

Robust spectrally-accurate bands via small \# grid evals
e.g. \( 25 \times 24 \times 24 \) for \( \omega \in [4, 6] \) and whole Brillouin zone, error \( 10^{-8} \)
Band structure to spectral accuracy

\( n = 0.3 \) inside
\( n = 1 \) outside
large inclusion

eval only \( 24 \times 24 \) samples in \( k \)
but contains much finer details
\( 10^{-8} \) errors, 1 hour on laptop

- Note: eigenvalues \( \omega_j(k) \) are not analytic!
  \( \exists \) conical (diabolical) points ... interpolates poorly
- like level set method: handle smooth func

movie 1
Software environment (teaser)
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MPSpack: object-oriented 2D PDE toolbox in Matlab (B-Betcke ’09)

- implements above & more: Helmholtz, Laplace, scattering
- intuitive interface: curves, domains, basis sets, problems, are objects

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E.g.

\[
\begin{align*}
s &= \text{segment.radialfunc}(100, \{ @(q)1 + 0.3\cos(3\pi q), @(q) - 0.9\sin(3\pi q) \})
\end{align*}
\]

test piecewise analytic curves
Software environment (teaser)

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E.g.

```matlab
s = segment.radialfunc(100, {@(q)1+.3*cos(3*q), @(q)-.9*sin(3*q)});

make exterior domain using segment
```
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    @(q)1+.3*cos(3*q),
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piecewise analytic curves

```matlab
d = domain([], [], s, -1);
```

make exterior domain using segment
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d.setbc(1, ’N’, []);
Neumann BCs on exterior
Software environment (teaser)

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```matlab
d.addmfsbasis(s, 200, struct('tau', 0.05));
```

choose basis set for solution
Software environment (teaser)

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Neumann BCs on exterior

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choose basis set for solution
```

(output of d.plot;
---
output of d.plot;
d.showbasesgeom;
\( p = \text{scattering}(d, \ [\]) \);  
make a scattering problem from domain \( d \)
p = scattering(d, []);

make a scattering problem from domain d

p.setoverallwavenumber(30); p.setincidentwave(pi/6);
p = scattering(d, []);

make a scattering problem from domain d

p.setoverallwavenumber(30); p.setincidentwave(pi/6);

p.solvecoeffs; p.bcresidualnorm

fills matrix, solves in 0.1 sec, $L^2$ error $6 \times 10^{-9}$
p = scattering(d, []);

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fills matrix, solves in 0.1 sec, $L^2$ error $6 \times 10^{-9}$

p.showfullfield(struct('bb', [-2 2 -2 2]));
p = scattering(d, []);

make a scattering problem from domain d

p.setoverallwavenumber(30); p.setincidentwave(pi/6);

p.solvecoeffs; p.bcresidualnorm

fills matrix, solves in 0.1 sec, $L^2$ error $6 \times 10^{-9}$

p.showfullfield(struct('bb',[-2 2 -2 2]));

- was easy case: 8 lines (could have done in 80 lines of Matlab)
- multiple (sub)domains: basis, quadrature, bookkeeping hidden
e.g. dielectric band structure still only 20 lines of code
- human-readable, rapid to code, sensible defaults (you can change)

To do: automatic meshing, Dirichlet EVP, …
Current work: grating scattering

Quasi-periodize in $x$-direction only: layer-potentials infinite in $y$...

- $N = 50$ unknowns on inclusion, $M = 200$ unknowns to periodize
- accuracy $10^{-13}$
Conclusions

- efficient 2nd-kind integral equations for photonic crystal EVP
- periodize via small # extra degrees of freedom on cell walls
- more robust and flexible than quasi-periodic Greens function:
  - no spurious blow-up at empty resonances
  - extends simply to 3D (unlike lattice sums)
- interpolate Fredholm det, not Bloch eigenvalues themselves

Future:

- 3D; drop in FMM for inclusion; gratings with substrate . . .

code:
http://code.google.com/p/mpspack

funding:
NSF DMS-0507614
DMS-0811005

Preprints, talks, movies:
http://math.dartmouth.edu/~ahb

made with: Linux, \LaTeX, Prosper
EXTRA SLIDES
Equal-frequency curves

Complexity of const-$\omega$ slice across Brillouin zone:

- only $24 \times 24$ evaluations of det $M$, Boyd’s spectral rootfinding
- Apps: Snell’s Law for reflection off semi-$\infty$ crystal