Dartmouth College Department of Mathematics
Math 101 Topics in Algebra: Quadratic Forms
Fall 2020

Final Exam (due Friday December 4)

Problems:

1. *Prime ideals in the Witt ring.* Let F be a field of characteristic not 2 and P a prime ideal of the Witt ring $W(F)$. Prove the following.

(a) If $W(F)/P$ has characteristic zero, then $W(F)/P \cong \mathbb{Z}$.

(b) If $W(F)/P$ has characteristic $p > 0$, then $W(F)/P \cong \mathbb{Z}/p\mathbb{Z}$.

(c) If $W(F)/P$ has characteristic 2, then $P = I$ is the fundamental ideal.

(d) If P is not the fundamental ideal, then $\{a \in F \mid \langle a \rangle \equiv 1 \pmod{P}\}$ is the set of positive elements of an ordering on F. If you don't know it, go look up what an *ordered field* is.

Hint. For any $a \in F^\times$, prove the relation $(\langle a \rangle + 1)(\langle a \rangle - 1) = 0$ in $W(F)$.

2. *Using Hasse–Minkowski.*

(a) Prove that the quadratic forms $\langle -1, 3, 5 \rangle$ and $\langle 1, 7, -105 \rangle$ are isometric over $\mathbb{Q}$.

(b) Which of the quadratic forms $\langle 1, -2, 5 \rangle$, $\langle 1, -1, 10 \rangle$, and $\langle 3, -1, 30 \rangle$ are isometric over $\mathbb{Q}$?

(c) Show that $\langle 1, 2, 5, -10 \rangle$ is anisotropic and universal (i.e., represents all nonzero values) over $\mathbb{Q}$.

(d) Show that $\langle 1, 1, 1, 7 \rangle$ is isotropic over $\mathbb{Q}_p$ for every prime p but is anisotropic over $\mathbb{R}$.

3. *An isotropic form.* Let p, q, r, s be distinct odd prime numbers such that $pqrs \not\equiv 1 \pmod{8}$. Prove that $\langle p, q, -r, -s \rangle$ is isotropic over $\mathbb{Q}$. **Hint.** The unique nonsplit quaternion algebra over $\mathbb{Q}_2$ is $(2, 5) = (-1, -1) = (-1, -5)$.

4. *Legendre.* Let a, b, c be nonzero squarefree integers that are pairwise relatively prime and not all of the same sign. Prove that $\langle a, b, c \rangle$ is isotropic over $\mathbb{Q}$ if and only if $-bc$ is a square modulo a, $-ac$ is a square modulo b, and $-ab$ is a square modulo c.

Historical note. That these conditions are necessary was discovered by Euler; Lagrange studied the case $a = 1$; Legendre proved that the conditions are sufficient; Gauss gave a second proof.