Dartmouth College Department of Mathematics
Math 101 Topics in Algebra: Quadratic Forms
Fall 2020

Problem Set # 2 (due Friday October 30)

Problems:

1. Let F be a finite field of odd order q and $W(F)$ be the Witt ring of quadratic forms over F. Prove that as rings $W(F) \cong \mathbb{Z}/4\mathbb{Z}$ if $q \equiv 3 \pmod 4$ and $W(F) \cong \mathbb{Z}/2\mathbb{Z}[F^\times/F^{\times 2}]$ if $q \equiv 1 \pmod 4$, where $\mathbb{Z}/2\mathbb{Z}[F^\times/F^{\times 2}] \cong \mathbb{Z}/2\mathbb{Z}[u]/(u^2-1)$ is the group ring of $F^\times/F^{\times 2}$ over $\mathbb{Z}/2\mathbb{Z}$.

2. Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathbb{C} \times \mathbb{C}$ as $\mathbb{R}$-algebras. Are they isomorphic as $\mathbb{C}$-algebras? **Hint.** Recall that an idempotent in a ring R is an element $e \in R$ such that $e^2 = e$. Given an idempotent $e \neq 1$ and letting $f = 1 - e$, we have a product decomposition $R \cong Re \times Rf$.

3. Prove that if F is an algebraically closed field (i.e., every nonconstant polynomial over F has a root), then the only finite dimensional division F-algebra is F itself. **Hint.** Let A be any finite dimensional F-algebra and prove that the linear map $l_x : A \to A$ defined by left multiplication by $x \in A \smallsetminus F$ has an eigenvalue $\lambda \in F$, and that $x - \lambda 1_D$ is a zero-divisor in A.

4. Let W be an F-vector space of dimension m. We defined the exterior algebra of W to be $\bigwedge W = T(W)/I$ where I is the two-sided ideal generated by $w \otimes w$ for all $w \in W$. The product of $\alpha, \beta \in \bigwedge W$ is denoted $\alpha \wedge \beta \in \bigwedge W$ and is often called the exterior product.

(a) Prove that $\bigwedge W$ is a graded algebra. The graded pieces are the exterior powers $\bigwedge^k W$ of W

$$\bigwedge W = \bigoplus_{k \geq 0} \bigwedge^k W$$

where $\bigwedge^0 W = F$ and $\bigwedge^1 W = W$.

(b) Prove that the map $\bigwedge^k (W^\vee) \times \bigwedge^k W \to F$ defined by

$$(f_1 \wedge \cdots \wedge f_k, w_1 \wedge \cdots \wedge w_k) \mapsto \det(f_i(w_j))_{1 \leq i,j \leq k}$$

is a bilinear pairing. Prove that this pairing induces a linear map $\bigwedge^k(W^\vee) \to \left(\bigwedge^k W\right)^\vee$ that is an isomorphism. You'll need to prove this in conjunction with the next part.

(c) If $\boldsymbol{x} = \{x_1, \dots, x_m\}$ is a basis of W, prove that

$$\wedge^k \boldsymbol{x} = \{x_{i_1} \wedge \cdots \wedge x_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq m\}$$

is a basis of $\bigwedge^k W$. In particular, prove that $\bigwedge^k W$ has dimension $\binom{n}{k}$, in particular, is trivial for $k > m$ or $k < 0$. You'll need to prove this in conjection with the previous part. **Hint.** To prove that a finite set of vectors $w_1, \dots, w_m$ is linearly independent, why is it sufficient to find elements $g_1, \dots, g_m$ of the dual space such that $g_i(w_j) = \delta_{ij}$? To help with the notation, you can call $x_I = x_{i_1} \wedge \cdots \wedge x_{i_k}$ where $I = \{i_1, \dots, i_k\}$ is a k-element subset of $\{1, \dots, m\}$.

(d) Prove that if the dimension of W is ≤ 3, then $\alpha \wedge \alpha = 0$ for all $\alpha \in \bigwedge^k W$ with $k > 0$, however, if the dimension of W is ≥ 4, then there always exists $\alpha \in \bigwedge^k W$, $k > 0$, with $\alpha \wedge \alpha \neq 0$.

(e) Given a linear operator $\varphi : W \to W$, prove that the induced map $\bigwedge^m \varphi : \bigwedge^m W \to \bigwedge^m W$ is multiplication by a scalar. Convince yourself that this scalar is the determinant of φ. (Some authors use this as the definition of the determinant.)

(f) Prove that the map $\bigwedge^k W \times \bigwedge^{m-k} W \to \bigwedge^m W$ defined by the exterior product is bilinear pairing. Prove that this pairing induces a linear map $\bigwedge^k W \to \left(\bigwedge^{m-k} W\right)^\vee \otimes \bigwedge^m W$ that is an isomorphism.

5. Let W be a finite dimensional F-vector space and let $V = W^\vee \oplus W$.

(a) Prove that the map $h : V \to F$ defined by $(f, w) \mapsto f(w)$ is a hyperbolic quadratic form.

(b) For any $f \in W^\vee$ and any $k \in \mathbb{Z}$ define the *contraction* map $d_f : \bigwedge^k W \to \bigwedge^{k-1} W$ by

$$d_f(w_1 \wedge \ldots \wedge w_k) = \sum_{j=1}^{k} (-1)^{j+1} f(w_j) w_1 \wedge \cdots \wedge w_{j-1} \wedge w_{j+1} \wedge \cdots \wedge w_k.$$

Prove that the family of maps $d_f : \bigwedge^k W \to \bigwedge^{k-1} W$, ranging over $k \in \mathbb{Z}$, are uniquely determined by the properties that $d_f(w) = f(w)$ for $w \in W = \bigwedge^1 W$ and

$$d_f(\alpha \wedge \beta) = d_f(\alpha) \wedge \beta + (-1)^a \alpha \wedge d_f(\beta)$$

for $\alpha \in \bigwedge^a W$ and $\beta \in \bigwedge^b W$. This is sometimes called a *derivation* of degree -1.

(c) For any $w \in W$ and any $k \in \mathbb{Z}$ define the map $l_w : \bigwedge^k W \to \bigwedge^{k+1} W$ by $l_w(\alpha) = w \wedge \alpha$. For any $w \in W$ and $f \in W^\vee$ prove that $l_w \circ l_w = 0$, $d_f \circ d_f = 0$, and that $l_w \circ d_f + d_f \circ l_w$ is multiplication by $f(w)$.

(d) Prove that the map $L : V \to \operatorname{End}_F(\bigwedge W)$ defined by $L(f, w) = d_f + l_w$ is F-linear, and that it induces, via the universal property of the Clifford algebra, an F-algebra isomorphism $C(V, h) \to \operatorname{End}_F(\bigwedge W)$.

(e) Let $E = E_0 \oplus E_1$ be a direct sum decomposition of a finite dimensional F-vector space E. Understand why we can consider elements of $\operatorname{End}_F(E)$ as "2×2 matrices"

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a \in \operatorname{End}_F(P_0), \quad b \in \operatorname{Hom}_F(P_1, P_0), \quad c \in \operatorname{Hom}_F(P_0, P_1), \quad d \in \operatorname{End}_F(P_1).$$

We can then consider the decomposition $\operatorname{End}_F(E) = \operatorname{End}_F(E)_0 \oplus \operatorname{End}_F(E)_1$, where $\operatorname{End}_F(E)_0$ is the subspace of "diagonal matrices," and $\operatorname{End}_F(E)_1$ is the subspace of "off-diagonal matrices," i.e., having the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \qquad \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix},$$

respectively. Prove that this decomposition gives $\operatorname{End}_F(E)$ the structure of a $\mathbb{Z}/2\mathbb{Z}$-graded F-algebra. This is called the *checkerboard grading.*

(f) Consider the decomposition $\bigwedge W = \bigwedge^{\text{even}} W \oplus \bigwedge^{\text{odd}} W$, where

$$\bigwedge^{\text{even}} W = \bigoplus_{k \text{ even}} \bigwedge^k W, \qquad \bigwedge^{\text{odd}} W = \bigoplus_{k \text{ odd}} \bigwedge^k W.$$

Prove that the F-algebra isomorphism $C(V, h) \to \operatorname{End}_F(\bigwedge W)$ considered above is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded algebras if we consider $\operatorname{End}_F(\bigwedge W)$ with the checkerboard grading determined by the decomposition $\bigwedge W = \bigwedge^{\text{even}} W \oplus \bigwedge^{\text{odd}} W$.

(g) Prove that there is an F-algebra isomorphism $C_0(V, h) \cong \operatorname{End}_F(\bigwedge^{\text{even}} W) \times \operatorname{End}_F(\bigwedge^{\text{odd}} W)$ and that the center of $C_0(V, h)$ is isomorphic to $F \times F$.