Dartmouth College Department of Mathematics
Math 101 Linear and Multilinear Algebra
Fall 2024

Problem Set # 1 (due via Canvas upload by 5 pm, Monday, September 30)

1. In this exercise, we prove (a more general version of) the exchange theorem: we will show that if β is a basis for a k-vector space V (not necessarily finite-dimensional) and $\{w_1, \dots, w_n\} \subseteq V$ is a linearly independent subset, then there exists a basis β' for V obtained by exchanging n vectors in β for the set $\{w_1, \dots, w_n\}$. In particular, $n \le |\beta|$.

(a) For the base case $n = 1$, since β is a basis we can write $w_1 = a_1 v_1 + \cdots + a_m v_m$ with $a_i \in k$ and $v_i \in \beta$ distinct. Since $w_1 \neq 0$, reordering we may suppose $a_1 \neq 0$. Argue that we can swap v_1 and w_1, i.e., $\beta' = \{w_1\} \cup (\beta \smallsetminus \{v_1\})$ is a basis for V.

(b) Finish the proof, using induction on n.

2. Let k be a finite field with $|k| = q$.

(a) By considering linear independence of columns, quickly recall that

$$|\operatorname{GL}_n(k)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$$

(see Dummit–Foote, p. 412).

(b) Let V be a k-vector space with $\dim_k V = 2$. What is the probability that an invertible endomorphism of V fixes at least one nonzero vector (i.e., $\phi(x) = x$ for some nonzero $x \in V$)?

3. For a set X recall the notion of the free vector space $k\langle X \rangle$ over a field k and its map $\iota_X : X \to k\langle X \rangle$.

(a) Let $h\colon X \to Y$ be map of sets. Show that there is a unique linear map $\phi\colon k\langle X \rangle \to k\langle Y \rangle$ so that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ h\ } & Y \\ \downarrow{\scriptstyle \iota_X} & & \downarrow{\scriptstyle \iota_Y} \\ k\langle X \rangle & \overset{\phi}{\dashrightarrow} & k\langle Y \rangle \end{array}$$

commutes.

(b) Write $h_* = \phi$ in the above. Suppose that $j\colon Y \to Z$ is another map of sets. Prove that $(j \circ h)_* = j_* \circ h_*$.

(c) Let $\{X_i\}_{i \in I}$ be sets indexed by a set I. Prove that there is an isomorphism

$$k \left\langle \textstyle\coprod_{i \in I} X_i \right\rangle \cong \bigoplus_{i \in I} k\langle X_i \rangle$$

4. Let V be a k-vector space. A map $\phi \in \operatorname{End}(V)$ is called a projection if $\phi^2 = \phi$. Let $\phi, \psi\colon V \to V$ be projection maps.

(a) Show that $V = \ker \phi \oplus \operatorname{img} \phi$.

(b) Suppose $\operatorname{char} k \neq 2$. Show that $\phi + \psi$ is a projection if and only if $\phi\psi = \psi\phi = 0$ if and only if $\operatorname{img} \phi \subseteq \ker \psi$ and $\operatorname{img} \psi \subseteq \ker \phi$.

(c) Still with $\operatorname{char} k \neq 2$, if $\phi + \psi$ is a projection, show that $\operatorname{img}(\phi + \psi) = \operatorname{img}(\phi) \oplus \operatorname{img}(\psi)$ and $\ker(\phi + \psi) = \ker(\phi) \cap \ker(\psi)$.

Hint. FIS 2.3 exercises 16 and (the hint in) 17 will come in handy.

5. Let V, W be finite-dimensional vector spaces over a field k, and let $\phi\colon V \to W$ be a k-linear map.

(a) Show that there exists a basis β of V and a basis γ of W such that $[\phi]_\beta^\gamma = (a_{ij})_{i,j}$ has $a_{ij} = 0$ unless $i = j$ and $a_{ii} \in \{0, 1\}$ (so the only nonzero entries are along the 'diagonal'). What does the number of 1s tell you about ϕ?

(b) Suppose now that $W = V$. Show that there exists a basis β of V such that the conclusion of (a) holds for $[\phi]_\beta$ if and only if ϕ is a projection.

6. Let $\{W_i\}_{i\in I}$ be subspaces, indexed by a set I, of a k-vector space V.

(a) Extend the definition of the sum of subspaces $\sum_{i\in I} W_i \subset V$ to an indexed set of subspaces.

(b) Consider the map

$$\Sigma : \bigoplus_{i\in I} W_i \to \sum_{i\in I} W_i$$

defined by $\Sigma(w_i) = \sum_{i\in I} w_i$, which makes sense since all but finitely many of the w_i are zero. We are considering the external direct sum on the left, and the internal sum on the right, of the subspaces W_i. Verify that Σ is a linear map and is surjective. Prove that if I has cardinality 2, then the kernel of Σ is $\bigcap_{i\in I} W_i$, but explain why in general, if I has cardinality > 2, than the kernel of Σ can be larger.

(c) Prove that Σ is an isomorphism if and only if $W_i \cap \sum_{j\neq i} W_j = \{0\}$ for all $i \in I$. In this case, we call the subspaces W_i disjoint and the internal sum "direct."