Dartmouth College Department of Mathematics
Math 101 Linear and Multilinear Algebra
Fall 2024

Problem Set # 2 (due via Canvas upload by 5 pm, Monday, October 7)

1. Let V, W be finite-dimensional k-vector spaces, $X \subseteq W$ be a subspace, and $\phi : V \to W$ a linear map. Prove that

$$\dim \phi^{-1}(X) \geq \dim V - \dim W + \dim X.$$

Hint. Consider the projection.

2. Let V be a finite-dimensional k-vector space and let $\phi : V \to V$ be alinear endomorphism of V.

(a) Show that there exists $m > 0$ so that $\operatorname{img}(\phi^m) \cap \ker(\phi^m) = \{0\}$.

(b) Now suppose that $\phi^2 = 0$. Show that the rank of ϕ is at most $(\dim V)/2$, and that there is an ordered basis β for V such that $[\phi]_\beta$ has the block form

$$\begin{pmatrix} O & A \\ O & O \end{pmatrix}$$

i.e., has zeros in all blocks except possibly the upper right-hand corner. In such a block form, what are the sizes of the blocks?

3. Let $0 \to Z \xrightarrow{\psi} V \xrightarrow{\phi} W \to 0$ be a short exact sequence of k-vector spaces.

(a) Let $Z' := \psi(Z)$. Show that there is a bijection between splittings τ of the short exact sequence and complements to Z', as follows: given a splitting τ, we take $W' := \tau(W)$, and given W', we take $\tau = (\phi|_{W'})^{-1}$. **Hint.** Recall that every short exact sequence is isomorphic to a split exact sequence.

(b) A retraction (of the short exact sequence) is a k-linear map $\rho : V \to Z$ such that $\rho \circ \psi = \mathrm{id}_Z$. Show that there is a bijection between retractions and complements to Z'.

4. Let V and W_i for $i \in I$ be k-vector spaces. Given a map $\phi : V \to \prod_i W_i$, we can postcompose with the projection onto the W_i component to get a map $\phi_i : V \to W_i$. Show that this induces a k-vector space isomorphism

$$\operatorname{Hom}\Big(V, \prod_{i \in I} W_i\Big) \xrightarrow{\sim} \prod_{i \in I} \operatorname{Hom}(V, W_i).$$

5. Let $V_\bullet$ be a bounded complex of k-vector spaces, i.e., $V_n = 0$ for all but finitely many $n \in \mathbb{Z}$.

(a) Define the Euler characteristic of $V_\bullet$ as

$$\chi(V_\bullet) = \sum_{n \in \mathbb{Z}} (-1)^n \dim V_n.$$

If $V_\bullet$ is exact, prove that $\chi(V_\bullet) = 0$. **Hint.** Using induction and rank-nullity.

(b) No longer assuming that $V_\bullet$ is exact, come up with an formula for the Euler characteristic $\chi(V_\bullet)$ involving the dimensions of the homology groups $H_n(V_\bullet)$.

6. Let V be a k-vector space.

(a) Let $\phi : V \to V$ be a projection. Show that the dual map $\phi^\vee : V^\vee \to V^\vee$ is a projection, and that $\operatorname{img} \phi^\vee = \operatorname{ann}(\ker \phi)$ and $\ker \phi^\vee = \operatorname{ann}(\operatorname{img} \phi)$.

(b) Assume that V is finite dimension and let $W \subseteq V$ be an subspace. Recall that the evaluation map $\mathrm{ev}\colon V \xrightarrow{\sim} V^{\vee\vee}$ is an isomorphism. Show that

$$\operatorname{ann}(\operatorname{ann}(W)) = \mathrm{ev}(W) \subseteq V^{\vee\vee}.$$