Dartmouth College Department of Mathematics
Math 101 Linear and Multilinear Algebra
Fall 2024

Problem Set # 3 (due via Canvas upload by 5 pm, Monday, October 14)

1. Let V be a k-vector space of dimension n and let $b\colon V \times V \to k$ be a nondegenerate alternating bilinear form on V.

(a) Let $W \subseteq V$ be a subspace such that the restriction $b_W := b|_{W\times W}\colon W \times W \to k$ is nondegenerate. Show that $V = W \oplus W^{\perp}$.

(b) Show that $n = 2m$ must be even, and there exists a basis $\beta = \{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ for V such that $b(e_i, e_j) = b(f_i, f_j) = 0$ for all $i, j = 1, \ldots, m$ and $b(e_i, f_j) = 1, 0$ whether $i = j$ or not. Such a basis is often called a symplectic basis. **Hint.** Mimic the Gram–Schmidt procedure.

(c) What is the Gram matrix $[b]_\beta$ with respect to a symplectic basis, and what if $\operatorname{char} k = 2$?

2. A quadratic form on a finite dimensional k-vector space V is a map $q\colon V \to k$ such that:

(i) For all $\lambda \in k$ and $x \in V$ we have $q(\lambda x) = \lambda^2 q(x)$, and

(ii) the map

$$b = b_q\colon V \times V \to k$$
$$(x, y) \mapsto q(x + y) - q(x) - q(y)$$

is a (symmetric) bilinear form, often called the associated bilinear form to q. Note that the set of quadratic forms on V is a k-vector space under addition of maps. An isometry between quadratic forms q on V and q' on V' is a k-linear isomorphism $\phi : V \to V'$ such that $q'(\phi(x)) = q(x)$ for all $x \in V$, and we define the orthogonal group $\mathrm{O}(q)$ of a quadratic form as the group of self-isometries.

(a) The map $q \mapsto b_q$ from quadratic forms to symmetric bilinear forms is k-linear, and is an isomorphism (that preserves orthogonal groups) when $\operatorname{char} k \neq 2$. **Hint.** In the other direction, consider the map $b \mapsto q_b : x \mapsto b(x, x)$.

(b) When $\operatorname{char} k = 2$, show, by way of examples, that the map $q \mapsto b_q$ from quadratic forms to symmetric bilinear forms is neither injective nor surjective in general and it does not generally preserve orthogonal groups.

(c) When $\operatorname{char} k = 2$, show that b_q is, in fact, an alternating bilinear form, and show that the map $q \mapsto b_q$ from quadratic forms to alternating bilinear forms, is surjective.

(d) Over any k, show that the map $b \mapsto q_b$ from *all* bilinear forms to quadratic forms is a k-linear surjection, and its kernel is the subspace of alternating bilinear forms.

(e) When $\operatorname{char} k \neq 2$ prove that any quadratic form is isometric to a diagonal quadratic form $\sum a_i x_i^2$ with $a_i \in k$. Show, by way of example, that not every quadratic form is diagonalizable when $\operatorname{char} k = 2$.

3. Consider the surface in $\mathbb{R}^3$ defined by

$$q(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = 1.$$

(a) Show that any quadratic form over $\mathbb{R}$ is isometric to a diagonal quadratic form $\sum a_i x_i^2$ with $a_i \in \{0, \pm 1\}$. For the above quadratic form q , how many of each of -1, 0, and 1 are there?

(b) Determine if the surface is an ellipsoid (an ellipse rotated about one of its axes), a hyperboloid (a hyperbola rotated about one of its axes), or a paraboloid (a parabola rotated about its axis).

4. Suppose $\operatorname{char} k \neq 2$ and let b be a symmetric bilinear form on a finite dimensional k-vector space. We say that $v \in V$ is anisotropic for b if $b(v,v) \neq 0$. For $v \in V$ anisotropic, define the orthogonal reflection along v by

$$\tau_v \colon V \to V$$
$$\tau_v(x) = x - 2\frac{b(v,x)}{b(v,v)}v$$

(a) How does this relate to the orthogonal projection used in the Gram–Schmidt procedure?

(b) Show that $\tau_v(v) = -v$ and that τ_v is the identity when restricted to $v^\perp$. Show that $\tau_v \in \mathrm{O}(b)$.

(c) Let $V = \mathbb{R}^3$ and b be the standard dot product. For $v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ compute the matrix of τ_v with respect to the standard basis.

(d) Show that the set of reflections is closed under conjugation in $\mathrm{O}(b)$.

(e) If $x, y \in V$ have $b(x,x) = b(y,y) \neq 0$, show there exists $\phi \in \mathrm{O}(b)$ such that $\phi(x) = y$. What does this say about the orbits of $\mathrm{O}(b)$ acting on V? **Hint.** Reflect along either $v = x + y$ or $v = x - y$.

(f) Is there a theory of orthogonal reflections for quadratic forms in characteristic 2?

5. In this exercise, we explore adjoints beyond the case of symmetric bilinear forms. Let $b \colon V \times W \to k$ and $b' \colon V' \times W' \to k$ be k-bilinear. For a k-linear map $\phi \colon V \to V'$, a k-linear map $\phi^* \colon W' \to W$ is said to be adjoint to ϕ (with respect to b and b') if

$$b(v, \phi^*(w')) = b'(\phi(v), w')$$

for all $v \in V$ and $w' \in W'$.

(a) Recall the tautological bilinear form

$$e \colon V \times V^\vee \to k$$
$$(v, f) \mapsto f(v),$$

defining e' similarly for V'. Show that $\phi^* = \phi^\vee$, i.e., the adjoint with respect to the tautological bilinear form is the dual.

(b) Show that if b is right nondegenerate, then an adjoint is unique.

(c) Show that if b is right nondegenerate and $\dim V = \dim W < \infty$, then a (unique) adjoint exists.