Dartmouth College Department of Mathematics
Math 101 Linear and Multilinear Algebra
Fall 2024

Problem Set # 5 (due via Canvas upload by 5 pm, Monday, October 28)

1. Let V, W, X be k-vector spaces.

(a) For k-vector spaces V' and W' and k-linear maps $\alpha : V' \to V$ and $\beta : W \to W'$, show that the map $\mathrm{Hom}(\alpha, \beta) : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V', W')$, defined by $\phi \mapsto \beta \circ \phi \circ \alpha$, is a k-linear map.

(b) Prove that there is a unique isomorphism

$$\mathrm{Hom}(V \otimes X, W) \xrightarrow{\sim} \mathrm{Hom}(V, \mathrm{Hom}(X, W))$$

with the property that $\phi \mapsto (v \mapsto (x \mapsto \phi(v \otimes x)))$. This is called the tensor-hom adjunction isomorphism. **Hint.** Under the isomorphism $\mathrm{Bil}_k(V, X; W) \cong \mathrm{Hom}(V \otimes X, W)$ the map is just taking the left polar.

(c) Prove that the tensor-hom adjunction isomorphism is "natural" i.e., given k-vector spaces V', W', X' and k-linear maps $\alpha\colon V' \to V$, $\beta\colon W \to W'$, and $\gamma\colon X' \to X$, the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{Hom}(V \otimes X, W) & \xrightarrow{\sim} & \mathrm{Hom}(V, \mathrm{Hom}(X, W)) \\
{\scriptstyle \mathrm{Hom}(\alpha\otimes\gamma,\beta)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(\alpha,\mathrm{Hom}(\gamma,\beta))} \\
\mathrm{Hom}(V' \otimes X', W') & \xrightarrow{\sim} & \mathrm{Hom}(V', \mathrm{Hom}(X', W'))
\end{array}$$

(d) Recall the k-linear map

$$\mathrm{Hom}(V, W) \otimes X \xhookrightarrow{(\sim)} \mathrm{Hom}(V, W \otimes X)\ ,$$

determined by $\phi \otimes x \mapsto (v \mapsto \phi(v) \otimes x)$, which is injective and is an isomorphism whenever V, W, X are finite-dimensional. This is called the hom-tensor adjunction map. Prove that the hom-tensor adjunction map is "natural", i.e., given k-vector spaces V', W', X' and k-linear maps $\alpha\colon V' \to V$, $\beta\colon W \to W'$, and $\gamma\colon X \to X'$, the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{Hom}(V, W) \otimes X & \xhookrightarrow{(\sim)} & \mathrm{Hom}(V, W \otimes X) \\
{\scriptstyle \mathrm{Hom}(\alpha,\beta)\otimes\gamma}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(\alpha,\beta\otimes\gamma)} \\
\mathrm{Hom}(V', W') \otimes X' & \xhookrightarrow{(\sim)} & \mathrm{Hom}(V', W' \otimes X')
\end{array}$$

So apparently tensor-hom adjunction is better-behaved than hom-tensor adjunction?

2. Let V be an k-vector space and let $d \geq 1$. The symmetric power S^dV is defined as a quotient of the tensor power $V^{\otimes d}$, from which it acquires an appropriate universal property. We can also work with a *subspace of symmetric tensors.* For each $\sigma \in \mathfrak{S}_d$, where $\mathfrak{S}_d$ is the symmetric group on $\{1, \dots, d\}$, we have an induced k-linear map $\sigma : V^{\otimes d} \to V^{\otimes d}$ determined on simple tensors by

$$\sigma(v_1 \otimes \cdots \otimes v_d) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)}$$

obtained by permuting coordinates. A tensor $\alpha \in V^{\otimes d}$ is called symmetric if $\sigma(\alpha) = \alpha$ for all $\sigma \in \mathfrak{S}_d$. Write $S_dV \subseteq V^{\otimes d}$ for the k-subspace of symmetric tensors.

(a) Suppose that $d! \in k^\times$, i.e., that $\mathrm{char}(k) > d$. Show that the map

$$\begin{aligned} s\colon S^d(V) &\to S_d(V) \\ \alpha &\mapsto \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \sigma(\alpha) \end{aligned}$$

is a well-defined k-linear isomorphism.

(b) Formulate and prove a similar statement for the subspace of skew-symmetric tensors, which satisfy $\sigma(\alpha) = (\mathrm{sgn}\,\sigma)\alpha$ for all $\sigma \in \mathfrak{S}_d$.

3. Let V be an k-vector space with $\dim V = n$, and let $\phi\colon V \to V$ be k-linear. Recall that we defined a unique determinant map $\det\colon V^n \to k$ (as a normalized, multilinear alternating form), and that we define $\det(\phi)$ by choosing a basis $v_1, \dots, v_n$ for V and taking $\det(\phi)\colon \det(\phi(v_1), \dots, \phi(v_n))$ (independent of the choice of basis, by uniqueness).

(a) Observe that ϕ induces a map $\bigwedge^n \phi\colon \bigwedge^n V \to \bigwedge^n V$ defined by $v_1 \wedge \cdots \wedge v_n \mapsto \phi(v_1) \wedge \cdots \wedge \phi(v_n)$.

(b) Recalling that $\dim(\bigwedge^n V) = 1$, explain why $\mathrm{Hom}(\bigwedge^n V, \bigwedge^n V) \cong k$ and $\bigwedge^n \phi$ is multiplication by some $d(\phi) \in k$.

(c) Show that $d(\phi) = \det(\phi)$. **Hint.** Show that the map $D\colon V^n \to k$ which starts with $(x_1, \dots, x_n) \in V^n$, makes the linear map $\phi \in \mathrm{End}(V)$ by $\phi(v_i) = x_i$, and then associates $d(\phi) \in k$, is a determinant function!

(d) Prove that $\det(\psi \circ \phi) = \det(\psi)\det(\phi) = \det(\phi \circ \psi)$ for all $\psi\colon V \to V$.

4. Let V be a k-vector space of finite dimension n.

(a) Prove that the k-bilinear map $\bigwedge^i V \times \bigwedge^{n-i} V \to \bigwedge^n V$, defined on simple wedges by the "wedging" $(v_1 \wedge \cdots \wedge v_i, v_{i+1} \wedge \cdots \wedge v_n) \mapsto v_1 \wedge \cdots \wedge v_n$, induces an isomorphism $\bigwedge^i V \cong (\bigwedge^{n-i} V)^\vee \otimes_k \bigwedge^n V$. What combinatorial formula do you derive by computing the dimensions of both sides?

(b) Let $0 \to W \xrightarrow{\phi} V \xrightarrow{f} L \to 0$ be a short exact sequence of k-vector spaces with L being 1-dimensional. Prove that the sequence

$$0 \to \textstyle\bigwedge^i W \xrightarrow{\wedge^i \phi} \bigwedge^i V \xrightarrow{d_f} L \otimes \bigwedge^{i-1} W \to 0$$

is short exact, where $d_f : \bigwedge^i V \to L \otimes \bigwedge^{i-1} W$ is the k-linear map uniquely determined on simple wedges by

$$d_f(v_1 \wedge \cdots \wedge v_i) = \sum_{j=1}^{i} (-1)^j f(v_j) \otimes v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_i,$$

where $v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_i$ means the wedge monomial obtained by omitting the jth term. What combinatorial formula do you derive by computing the dimensions via rank-nullity?

5. Recall that if A, B are k-algebras (with 1) then there is a unique structure of k-algebra on $A \otimes_k B$ with the property that

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$$

for all $a, a' \in A$ and $b, b' \in B$. Furthermore, if $k \subset K$ is a field extension, recall the extension of scalars $A_K = A \otimes_k K$, which has a unique structure of K-algebra with the property that

$$\lambda \cdot (a \otimes \alpha) = a \otimes \lambda\alpha$$

for all $a \in A$ and $\lambda, \alpha \in K$. For a K-algebra B, recall the restriction of scalars ${}_kB$, which is a k-algebra by restricting the scalar multiplication from K to k.

(a) For k-algebras A and B, verify (whenever it makes sense) that

$$\dim_k(A \otimes_k B) = \dim_k A \cdot \dim_k B$$

(b) For a k-algebra A and a K-algebra B, verify (whenever they make sense) that

$$\dim_K(A_K) = \dim_k A, \qquad \dim_k({}_kB) = \dim_k K \cdot \dim_K B$$

(c) Consider the $\mathbb{R}$-algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}$ as $\mathbb{R}$-algebras. **Hint.** Define the map $\Phi\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $(z, w) \mapsto (zw, z\overline{w})$ and use it to get an $\mathbb{R}$-algebra isomorphism $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \times \mathbb{C}$.

(d) Let $\sigma : K \to K$ be a k-algebra automorphism and W a K-vector space. Define a new K-vector space ${}^\sigma W$ on the same underlying abelian group W with scalar multiplication $\lambda \cdot w = \sigma(\lambda)w$ for all $\lambda \in K$ and $w \in W$. This is called the σ-twist of W. Prove that $\dim_K({}^\sigma W) = \dim_K W$. Prove that to give a K-linear map $\phi : W \to {}^\sigma W'$ is the same as to give a k-linear map $\phi : {}_kW \to {}_kW'$ such that $\phi(\lambda w) = \sigma(\lambda)\phi(w)$ for all $\lambda \in K$ and $w \in W$. Such ϕ are often called σ-semilinear.

(e) When $k = \mathbb{R}$, $K = \mathbb{C}$, and σ is complex conjugation, ${}^\sigma W$ is often denoted by $\overline{W}$. Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \overline{\mathbb{C}}$ as $\mathbb{C}$-algebras.