Dartmouth College Department of Mathematics
Math 101 Linear and Multilinear Algebra
Fall 2024

Problem Set # 6 (due via Canvas upload by 5 pm, Friday, November 8)

1. To prove that a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are an adjoint pair, it suffices to construct natural transformations $\eta : I_{\mathcal{C}} \to G \circ F$ and $\varepsilon : F \circ G \to I_{\mathcal{D}}$ called the unit and counit of adjunction such that the following counit-unit equations are satisfies: for every $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$ we have

$$\mathrm{id}_{F(X)} = \varepsilon_{F(X)} \circ F(\eta_X), \qquad \mathrm{id}_{G(Y)} = G(\varepsilon_Y) \circ \eta_{G(Y)}.$$

(a) Practice constructing the unit and counit of adjunction and using the counit-unit equations to prove that the free vector space functor $F\langle - \rangle : \textsf{Set} \to \textsf{Vect}_F$ is left adjoint to the forgetful functor $\textsf{Vect}_F \to \textsf{Set}$.

(b) Let K/k be a field extension. Prove that the extension of scalars functor $(-)_K : k\text{-}\textsf{Alg} \to K\text{-}\textsf{Alg}$ is left adjoint to the restriction of scalars functor ${}_k(-) : K\text{-}\textsf{Alg} \to k\text{-}\textsf{Alg}$.

(c) Let G be a group and $H \subset G$ be a subgroup. Recall from the daily homework that the category $\textsf{Rep}_G$ of representations of G is isomorphic to the category $\mathbb{Z}\langle G\rangle\text{-}\textsf{Mod}$ of modules over the group ring. The restriction functor $\mathrm{res}^G_H : \textsf{Rep}_G \to \textsf{Rep}_H$ is defined by simply viewing a G-representation as an H-representation by restricting the action, and similarly on morphisms. The induction functor $\mathrm{ind}^G_H : \textsf{Rep}_H \to \textsf{Rep}_G$ is defined, for an H-representation W, as

$$\mathrm{ind}^G_H(W) = \{f : G \to W \ : \ f(gh^{-1}) = h \cdot f(g) \text{ for all } g \in G, h \in H\}$$

where the G-action on $\mathrm{ind}^G_H(W)$ is $(g \cdot f)(x) = f(g^{-1}x)$ for $g \in G$ and $\mathrm{ind}^G_H(\phi)(f) = \phi \circ f$ on morphisms $\phi : W \to V$ between H-representations. Prove Frobenius reciprocity, the statement that ind^G_H if left adjoint to res^G_H. **Hint.** Use the fact that, via the isomorphism of categories $\textsf{Rep}_G \to \mathbb{Z}\langle G\rangle\text{-}\textsf{Mod}$ the induction functor is identified with the functor $M \mapsto \mathbb{Z}\langle G\rangle \otimes_{\mathbb{Z}\langle H\rangle} M$.

Do you see a similarity between the later two?

2. Let R be a commutative ring. Let J be an ideal of R.

(a) For M a R-module, let

$$JM = \left\{ \sum_{i=1}^{n} a_i x_i : a_i \in J, x_i \in M \right\}.$$

Show that JM is an R-submodule of M.

(b) If $\phi\colon M \to N$ is an isomorphism of R-modules, show that $\phi|_{JM}$ induces an isomorphism $JM \simeq JN$.

(c) Let $\{M_i\}_{i\in I}$ be R-modules and let $N_i \subset M_i$ be R-submodules for all i. Prove that

$$\left(\bigoplus_{i\in I} M_i\right) \Big/ \left(\bigoplus_{i\in I} N_i\right) \simeq \bigoplus_{i\in I} M_i/N_i.$$

(d) If $M \simeq \bigoplus_{i\in I} R$ is a free R-module, show that

$$M/JM \simeq \bigoplus_{i\in I} R/J$$

as R-modules.

(e) Suppose that R is not the zero ring. Prove that two free R-modules are isomorphic if and only if they have R-bases of the same cardinality, in particular $R^n \simeq R^m$ if and only if $n = m$. **Hint.** Apply (d) with J a maximal ideal of R.

3. Let R be an integral domain and let M be a finitely generated R-module. The *rank* of M is the maximal number of R-linearly independent elements of M.

We already defined the rank of a free R-module with basis β to be $\#\beta$, and in the previous exercise we showed it is well-defined. But if we are going to use the same word *rank*, we should show that the two notions concide when M is free (over a domain)! (If this is confusing to you, use the term *free rank* for free modules while you work on this exercise, then afterwards you can drop the "free".)

(a) Suppose that M has rank n and that $x_1, \ldots, x_n$ is any maximal set of R-linearly independent elements of M. Let $N = Rx_1 + \cdots + Rx_n$ be the R-submodule generated by $x_1, \ldots, x_n$. Prove that N is isomorphic to R^n and that the quotient M/N is a torsion R-module. **Hint.** Show that the map $R^n \to N$ which sends the ith standard basis vector to x_i is an isomorphism of R-modules.

(b) Prove conversely that if M contains a submodule N that is free of rank n (i.e., $N \simeq R^n$) such that the quotient M/N is a torsion R-module then M has rank n. **Hint.** Let $y_1, \ldots, y_{n+1}$ be any $n+1$ elements of M. Use the fact that M/N is torsion to write $r_i y_i$ as a linear combination of a basis for N for some nonzero elements r_i of R. Show that the $r_i y_i$, and hence also the y_i, are linearly dependent, working over the field of fractions.

(c) Conclude that $M = R^n$ has rank n, i.e., that the maximum number of R-linear independent elements in R^n is n (the same as the cardinality of the standard basis).

(d) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated R-modules. Prove that the rank of M is the sum of the ranks of M' and M''.

4. Let R be a commutative ring. A unimodular row (of length n over R) is a row vector $a = (a_1, \ldots, a_n)$ such that $a_i \in R$ collectively generate R.

(a) Define a map $\phi_a : R^n \to R$ by $\phi_a(v) = av$, where av denotes the usual product of a row by a column to get an element of R. Prove that $P_a = \ker \phi_a$ is a projective R-module of rank $n-1$. **Hint.** Check that a is a unimodular row if and only if there exist $b_i \in R$ with $\sum_{i=1}^n a_i b_i = 1$.

(b) For unimodular rows a, b of length n over R, prove that $P_a \cong P_b$ if and only if there exists $G \in \mathrm{GL}_n(R)$ such that $aG = b$, in which case we write $a \sim b$. This determines an equivalence relation on unimodular rows.

(c) We say that a unimodular row a is trivial if $a \sim e_1$, where $e_1 \in R^n$ is the first standard basis vector. Prove that a unimodular row a is trivial if and only if P_a is a free R-module.

(d) A vector $v \in R^n$ is called completable if v can be completed to an R-basis of R^n. Prove that $v \in R^n$ is completable if and only if v is the first column of an element of $\mathrm{GL}_n(R)$ if and only if $v = a^\mathsf{T}$ for a trivial unimodular row a.

(e) Prove that any unimodular row (a_1, a_2) of length 2 over R is trivial. **Hint.** Consider $(-b_2, b_1)$.

(f) Prove that if R is a PID then every unimodular row is trivial.

The topology of the 2-sphere can be used to prove that over $R = \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$, the unimodular row (x, y, z) is not trivial. **History.** In 1976, Quillen and Suslin independently proved that any unimodular row over $k[x_1, \ldots, x_n]$ is trivial, settling a problem posed by Serre in 1955. Quillen's 1978 Fields Medal was given in part for this work.

5. Let V be a k-vector space of finite dimension n and $\phi\colon V \to V$ an k-linear endomorphism, which gives V the structure of a $k[x]$-module. The point of this problem is to describe this $k[x]$-module by generators and relations, i.e., as the quotient of a free module.

Fixing a basis $\beta = \{v_1, \ldots, v_n\}$ of V, we consider the surjective R-module homomorphism

$$\begin{aligned} \pi\colon k[x]\langle\beta\rangle &\to V \\ v_i &\mapsto v_i \end{aligned}$$

Then identifying $k[x]\langle\beta\rangle = k[x]^n$, we have an isomorphism of $k[x]$-modules $V \cong k[x]^n/\ker\pi$.

(a) Show that the elements $xv_i - \phi(v_i)$ for $i = 1, \ldots, n$ belong to $\ker\pi$. Here $\phi(v_i)$ is considered via the inclusion $V \cong k\langle\beta\rangle \hookrightarrow k[x]\langle\beta\rangle$.

(b) Let $N = \langle xv_i - \phi(v_i)\rangle_{i=1,\ldots,n}$ be the $k[x]$-submodule of $k[x]\langle\beta\rangle$ generated by the elements $xv_i - \phi(v_i)$ for $i = 1, \ldots, n$. Show that $N = \ker\pi$. **Hint.** Let $\sum_{i=1}^n f_i(x) v_i \in \ker\pi$. If all the $f_i(x) = c_i \in k$ are constant, conclude that $c_i = 0$ for all i. Finish by induction on the maximum degree of the $f_i(x)$ by writing $f_i(x) = x g_i(x) + c_i$ with $c_i \in F$, by noting that $\sum_{i=1}^n f_i(x) v_i - \sum_{i=1}^n c_i v_i - \sum_{i=1}^n g_i(x)\phi(v_i) \in N$.

(c) Conclude that the invariant factors of V as a $k[x]$-module are obtained from the Smith normal form of the relations matrix $[\phi - x\,\mathrm{id}_V]_\beta \in M_n(k[x])$.