

Problem Set # 1 (due via Canvas upload by 5 pm, Monday, September 29)

1. In this exercise, we prove (a more general version of) the exchange theorem: we will show that if β is a basis for a k -vector space V (not necessarily finite-dimensional) and $\{w_1, \dots, w_n\} \subseteq V$ is a linearly independent subset, then there exists a basis β' for V obtained by exchanging n vectors in β for the set $\{w_1, \dots, w_n\}$. In particular, $n \leq |\beta|$.

- (a) For the base case $n = 1$, since β is a basis we can write $w_1 = a_1v_1 + \dots + a_mv_m$ with $a_i \in k$ and $v_i \in \beta$ distinct. Since $w_1 \neq 0$, reordering we may suppose $a_1 \neq 0$. Argue that we can swap v_1 and w_1 , i.e., $\beta' = \{w_1\} \cup (\beta \setminus \{v_1\})$ is a basis for V .
- (b) Finish the proof, using induction on n .

2. Let k be a finite field with $|k| = q$.

- (a) By considering linear independence of columns, quickly recall that

$$|\mathrm{GL}_n(k)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$$

(see Dummit–Foote, p. 412).

- (b) Let V be a k -vector space with $\dim_k V = 2$. What is the probability that an invertible endomorphism of V fixes at least one nonzero vector, i.e., $\phi(x) = x$ for some nonzero $x \in V$?

3. For a set X recall the notion of the free vector space $k\langle X \rangle$ over a field k and its map $\iota_X : X \rightarrow k\langle X \rangle$.

- (a) Let $h : X \rightarrow Y$ be map of sets. Show that there is a unique linear map $h_* : k\langle X \rangle \rightarrow k\langle Y \rangle$ so that the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ \downarrow \iota_X & & \downarrow \iota_Y \\ k\langle X \rangle & \xrightarrow{h_*} & k\langle Y \rangle \end{array}$$

commutes.

- (b) Suppose that $j : Y \rightarrow Z$ is another map of sets. Prove that $(j \circ h)_* = j_* \circ h_*$.
- (c) Let $\{X_i\}_{i \in I}$ be sets indexed by a set I . Prove that there is an isomorphism

$$k\langle \bigsqcup_{i \in I} X_i \rangle \cong \bigoplus_{i \in I} k\langle X_i \rangle$$

4. Let V be a k -vector space. A map $\phi \in \mathrm{End}(V)$ is called a **projection** if $\phi^2 = \phi$. Let $\phi, \psi : V \rightarrow V$ be projection maps.

- (a) Show that $V = \ker \phi \oplus \mathrm{img} \phi$.
- (b) Suppose $\mathrm{char} k \neq 2$. Show that $\phi + \psi$ is a projection if and only if $\phi\psi = \psi\phi = 0$ if and only if $\mathrm{img} \phi \subseteq \ker \psi$ and $\mathrm{img} \psi \subseteq \ker \phi$.
- (c) Still with $\mathrm{char} k \neq 2$, if $\phi + \psi$ is a projection, show that $\mathrm{img}(\phi + \psi) = \mathrm{img}(\phi) \oplus \mathrm{img}(\psi)$ and $\ker(\phi + \psi) = \ker(\phi) \cap \ker(\psi)$.

Hint. FIS 2.3 exercises 16 and (the hint in) 17 will come in handy.

5. Let V, W be finite-dimensional vector spaces over a field k , and let $\phi: V \rightarrow W$ be a k -linear map.
- (a) Show that there exists a basis β of V and a basis γ of W such that $[\phi]_{\beta}^{\gamma} = (a_{ij})_{i,j}$ has $a_{ij} = 0$ unless $i = j$ and $a_{ii} \in \{0, 1\}$ (so the only nonzero entries are along the ‘diagonal’). What does the number of 1s tell you about ϕ ?
 - (b) Suppose now that $W = V$. Show that there exists a basis β of V such that the conclusion of (a) holds for $[\phi]_{\beta}$ if and only if ϕ is a projection.
6. Let $\{W_i\}_{i \in I}$ be subspaces, indexed by a set I , of a k -vector space V .
- (a) Extend the definition of the sum of subspaces $\sum_{i \in I} W_i \subset V$ to an indexed set of subspaces.
 - (b) Consider the map

$$\Sigma: \bigoplus_{i \in I} W_i \rightarrow \sum_{i \in I} W_i$$

defined by $\Sigma(w_i) = \sum_{i \in I} w_i$, which makes sense since all but finitely many of the w_i are zero. We are considering the external direct sum on the left, and the internal sum on the right, of the subspaces W_i . Verify that Σ is a linear map and is surjective. Prove that if I has cardinality 2, then the kernel of Σ is $\bigcap_{i \in I} W_i$, but explain why in general, if I has cardinality > 2 , then the kernel of Σ can be larger.

- (c) Prove that Σ is an isomorphism if and only if $W_i \cap \sum_{j \neq i} W_j = \{0\}$ for all $i \in I$. In this case, we call the subspaces W_i disjoint and the internal sum “direct.”