Dartmouth College Department of Mathematics
Math 101 Linear and Multilinear Algebra
Fall 2025

Problem Set # 3 (due via Canvas upload by 5 pm, Friday, October 17)

1. Let V be a F-vector space of dimension n and let $T\colon V \times V \to F$ be a nondegenerate alternating bilinear form on V.

(a) Let $W \subseteq V$ be a subspace such that the restriction $T_W := T|_{W\times W}\colon W \times W \to F$ is nondegenerate. Show that $V = W \oplus W^\perp$.

(b) Show that $n = 2m$ must be even, and there exists a basis $\beta = \{e_1, \dots, e_m, f_1, \dots, f_m\}$ for V such that $b(e_i, e_j) = b(f_i, f_j) = 0$ for all $i, j = 1, \dots, m$ and $b(e_i, f_j) = 1, 0$ whether $i = j$ or not. Such a basis is often called a symplectic basis. **Hint.** Mimic the Gram–Schmidt procedure.

(c) What is the Gram matrix $[T]_\beta$ with respect to a symplectic basis, and what if $\operatorname{char} k = 2$?

2. Let Q be a quadratic form on a finite dimensional F-vector space and recall the orthogonal group $\mathrm{O}(Q)$ of Q.

(a) Show that when $\operatorname{char} F \neq 2$, the map $Q \mapsto T_Q$ from quadratic forms to symmetric bilinear forms is a T-linear isomorphism that preserves orthogonal groups. **Hint.** In the other direction, consider the map $T \mapsto Q_T : x \mapsto T(x, x)$.

(b) When $\operatorname{char} F = 2$, show, by way of examples, that the map $Q \mapsto T_Q$ from quadratic forms to symmetric bilinear forms is neither injective nor surjective in general and it does not generally preserve orthogonal groups.

(c) When $\operatorname{char} F = 2$, show that T_Q is, in fact, an alternating bilinear form, and show that the map $Q \mapsto T_Q$ from quadratic forms to alternating bilinear forms, is surjective.

(d) Over any F, show that the map $T \mapsto Q_T$ from *all* bilinear forms to quadratic forms is a F-linear surjection, and its kernel is the subspace of alternating bilinear forms.

(e) Show, by way of example, that not every quadratic form is diagonalizable when $\operatorname{char} F = 2$.

3. Consider the surface in $\mathbb{R}^3$ defined by

$$Q(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = 1.$$

(a) Recall that any quadratic form over $\mathbb{R}$ is isometric to a diagonal quadratic form $\sum a_i x_i^2$ with $a_i \in \{0, \pm 1\}$. For the above quadratic form q , how many of each of -1, 0, and 1 are there?

(b) Determine if the surface is an ellipsoid (an ellipse rotated about one of its axes), a hyperboloid (a hyperbola rotated about one of its axes), or a paraboloid (a parabola rotated about its axis).

4. Suppose $\operatorname{char} F \neq 2$ and let T be a symmetric bilinear form on a finite dimensional F-vector space. We say that $v \in V$ is anisotropic for T if $T(v, v) \neq 0$. For $v \in V$ anisotropic, define the orthogonal reflection along v by

$$\tau_v\colon V \to V$$
$$\tau_v(x) = x - 2\frac{b(v, x)}{b(v, v)} v$$

(a) How does this relate to the orthogonal projection used in the Gram–Schmidt procedure?

(b) Show that $\tau_v(v) = -v$ and that τ_v is the identity when restricted to $v^\perp$. Show that $\tau_v \in \mathrm{O}(T)$.

(c) Let $V = \mathbb{R}^3$ and T be the standard dot product. For $v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ compute the matrix of τ_v with respect to the standard basis.

(d) Show that the set of reflections is closed under conjugation in $\mathrm{O}(T)$.

(e) If $x, y \in V$ have $b(x,x) = b(y,y) \neq 0$, show there exists $\phi \in \mathrm{O}(T)$ such that $\phi(x) = y$. What does this say about the orbits of $\mathrm{O}(T)$ acting on V? **Hint.** Reflect along either $v = x + y$ or $v = x - y$.

(f) Is there a theory of orthogonal reflections for quadratic forms in characteristic 2?

5. For a nondegenerate quadratic form Q on an $\mathbb{R}$-vector space V of dimension n and signature s, compute the set of possible similitude factors of Q as a subset of $\mathbb{R}^\times$.

6. In this exercise, we explore adjoints beyond the case of symmetric bilinear forms. Let $T \colon V \times W \to k$ and $T' \colon V' \times W' \to F$ be F-bilinear maps. For an F-linear map $\phi \colon V \to V'$, an F-linear map $\phi^* \colon W' \to W$ is said to be adjoint to ϕ (with respect to T and T') if

$$T(v, \phi^*(w')) = T'(\phi(v), w')$$

for all $v \in V$ and $w' \in W'$.

(a) Recall the tautological bilinear form

$$\begin{aligned} e \colon V \times V^\vee &\to F \\ (v, f) &\mapsto f(v) \end{aligned}$$

defining e' similarly for V'. Show that $\phi^* = \phi^\vee$, i.e., the adjoint with respect to the tautological bilinear form is the dual.

(b) Show that if T is right nondegenerate, then an adjoint is unique.

(c) Show that if T is right nondegenerate and $\dim V = \dim W < \infty$, then a (unique) adjoint exists.