Dartmouth College Department of Mathematics
Math 101 Linear and Multilinear Algebra
Fall 2025

Problem Set # 4 (due via Canvas upload by 5 pm, Wednesday, October 29)

1. Let V, W be finite-dimensional complex inner product spaces.

(a) Let $\phi : V \to V$ be a self-adjoint linear operator. We say $\phi : V \to V$ is positive semidefinite if $\langle \phi(x), x\rangle \in \mathbb{R}_{\geq 0}$ for all $x \in V$. Show that ϕ is positive semidefinite if and only if all of the eigenvalues of ϕ are in $\mathbb{R}_{\geq 0}$.

(b) Now let $\phi : V \to W$ be a linear map. Show that $\phi^*\phi$ and $\phi\phi^*$ are (self-adjoint and) positive semidefinite.

(c) Continuing (b), show that $\mathrm{rk}(\phi^*\phi) = \mathrm{rk}(\phi\phi^*) = \mathrm{rk}(\phi)$. **Hint.** Show that $\ker \phi^*\phi = \ker \phi$.

(d) Finally, show that $\phi : V \to W$ admits a *singular value decomposition* as follows: there exist orthonormal bases $\beta = \{v_1, \ldots, v_n\}$ for V and $\gamma = \{w_1, \ldots, w_m\}$ for W, and real numbers

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

such that

$$\phi(v_i) = \begin{cases} \sigma_i w_i, & \text{if } 1 \leq i \leq r; \\ 0, & \text{if } i > r \end{cases}$$

Hint. Let $\lambda_1 \geq \cdots \geq \lambda_r > 0$ be the nonzero eigenvalues of $\phi^*\phi$ corresponding to an orthonormal basis $\beta = \{v_1, \ldots, v_n\}$; let $\sigma_i = \sqrt{\lambda_i} \geq 0$ and $w_i = \sigma_i^{-1}\phi(v_i)$ for $i = 1, \ldots, r$. Check that w_i are orthonormal, and extend to an orthonormal basis.

2. Let $V = F[x]$ be the F-vector space of polynomials in x. Exhibit an explicit isomorphism

$$V \otimes_k V \simeq F[x_1, x_2]$$

to the F-vector space of polynomials in two variables.

3. Let V, W be F-vector spaces. Assume that $v_1, \ldots, v_n \in V$ are linearly independent. Show that if $w_1, \ldots, w_n \in W$ with

$$v_1 \otimes w_1 + \cdots + v_n \otimes w_n = 0$$

then $w_1 = \cdots = w_n = 0$. In fact, prove this two ways:

(a) First, brutally argue with bases (extending to a basis for V and choosing a basis for W).

(b) Second, for each $g \in W^\vee$, consider the map $V \times W \to V$ by $(v, w) \mapsto vg(w)$, and apply the universal property.

Conclude that if $v \in V$ and $w \in W$ has $v \otimes w = 0$, then either $v = 0$ or $w = 0$.

4. In this problem, we will try answer the question: what is the probability that a tensor is a simple tensor? Let V and W be F-vector spaces with dimensions n and m, respectively.

(a) We first consider the case where F is a finite field with $\#F = q$. Show that the number of distinct *nonzero* simple tensors in $V \otimes_F W$ is $(q^n - 1)(q^m - 1)/(q - 1)$. Conclude that the probability that an element of $V \otimes W$ is a simple tensor is 1 if $m = 1$ or $n = 1$, and otherwise is approximately $1/q^{(n-1)(m-1)}$, in which case this probability converges to 0 as $q \to \infty$.

(b) Now let F be any field. Give a proof for (or recall) the fact that a matrix $A \in \mathrm{M}_{m,n}(F)$ has rank r (with $r \leq m$ and $r \leq n$) if and only if there exists a nonvanishing $r \times r$-minor (a determinant of a submatrix of A taking r distinct rows and r distinct columns) and all minors of size $> r$ are zero. **Hint.** Consider linear independence of columns or rows.

(c) Finally, we consider the situation over $F = \mathbb{R}$. Assume that $m, n \geq 2$. Show that in the isomorphism $\mathbb{R}^n \otimes \mathbb{R}^m \simeq \mathrm{M}_{m,n}(\mathbb{R})$, the image of the set of simple tensors corresponds to the fiber over zero under the map

$$\begin{aligned} \mathrm{M}_{m,n}(\mathbb{R}) &\to \mathbb{R}^{\binom{n}{2}\binom{m}{2}} \\ (a_{ij})_{i,j} &\mapsto (a_{ij}a_{i'j'} - a_{ij'}a_{i'j})_{i,j,i',j'} \end{aligned}$$

sending a matrix to the tuple of all of its 2×2 minors. Give a plausible reason or a rigorous argument (if you know some analysis) that this fiber has Lebesgue measure zero, so "a random tensor in $\mathbb{R}^n \otimes \mathbb{R}^m$ is a simple tensor with probability zero". **Hint.** Recall the following consequence of the inverse/implicit function theorem: if $\varphi : M \to N$ is a differentiable map of manifolds whose differential $D\varphi|_x$ has rank r for all $x \in \varphi^{-1}(y)$ for some $y \in N$, then $\varphi^{-1}(y) \subset M$ is a submanifold of codimentions r.

5. Let $V_1, \ldots, V_s$ be F-vector spaces of dimension $n_1, \ldots, n_s$. Recall the rank of a tensor $\alpha \in V = V_1 \otimes \cdots \otimes V_s$ to be the minimal $r \geq 0$ such that $\alpha = \sum_{i=1}^r v_1 \otimes \cdots \otimes v_s$ with $v_i \in V_i$. A tensor has rank 0 if and only if it's zero and has rank 1 if and only if it's simple.

(a) Prove that via the isomorphism $V_1^\vee \otimes V_2 \cong \mathrm{Hom}_F(V_1, V_2)$ the rank of a tensor equals the rank of the corresponding linear map.

(b) Prove that the rank of a tensor $\alpha \in V$ is bounded by $n_1 \cdots n_s / \max\{n_1, \ldots, n_s\}$. Explain why this generalizes the case for linear maps.

(c) In the case $r = 3$ and $n_1 = n_2 = n_3 = 2$, write a tensor of each possible rank.

6. Let V be a real inner product space with $\dim V = n$. Let

$$S = \{x \in V : \|x\|^2 = 1\}$$

be the $(n-1)$-dimensional unit sphere in V. Let $\phi : V \to V$ be a self-adjoint linear map (i.e., $\phi = \phi^*$). In this exercise, we give a self-contained "physical" or "geometric" proof of Cauchy's theorem that ϕ has an orthonormal basis of eigenvectors: we find an eigenvector by maximizing on the sphere!

(a) Suppose that $x, y \in S$ have $\langle x, y\rangle = 0$. Show that $\cos(t)x + \sin(t)y$ lies on S for all $t \in \mathbb{R}$.

(b) By vector calculus, the function $x \mapsto \langle x, \phi(x)\rangle$ achieves a maximum at some point $p \in S$: briefly explain why. Let $y \in S$ satisfy $\langle p, y\rangle = 0$. Consider the function

$$f(t) = \langle \cos(t)p + \sin(t)y, \phi(\cos(t)p + \sin(t)y)\rangle.$$

Show that $\langle p, \phi(y)\rangle = 0$.

(c) Let $W = \mathrm{span}(\{p\}) \subset V$. Show that $W^\perp$ is ϕ-invariant and then conclude that W is ϕ-invariant. Conclude that p is an eigenvector!

(d) Show by induction that V has an orthonormal basis of vectors that are eigenvectors for ϕ.