Dartmouth College Department of Mathematics
Math 101 Linear and Multilinear Algebra
Fall 2025

Problem Set # 5 (due via Canvas upload by 5 pm, Monday, November 10)

1. Let V be an F-vector space with $\dim V = n$, and let $\phi\colon V \to V$ be F-linear. Recall that we defined a unique determinant map $\det\colon V^n \to F$ (as a normalized, multilinear alternating form), and that we define $\det(\phi)$ by choosing a basis $v_1, \dots, v_n$ for V and taking $\det(\phi)\colon \det(\phi(v_1), \dots, \phi(v_n))$ (independent of the choice of basis, by uniqueness).

(a) Observe that ϕ induces a map $\bigwedge^n \phi\colon \bigwedge^n V \to \bigwedge^n V$ defined by $v_1 \wedge \cdots \wedge v_n \mapsto \phi(v_1) \wedge \cdots \wedge \phi(v_n)$.

(b) Recalling that $\dim(\bigwedge^n V) = 1$, explain why $\mathrm{Hom}(\bigwedge^n V, \bigwedge^n V) \cong F$ and $\bigwedge^n \phi$ is multiplication by some $d(\phi) \in F$.

(c) Show that $d(\phi) = \det(\phi)$. **Hint.** Show that the map $D\colon V^n \to F$ which starts with $(x_1, \dots, x_n) \in V^n$, makes the linear map $\phi \in \mathrm{End}(V)$ by $\phi(v_i) = x_i$, and then associates $d(\phi) \in F$, is a determinant function!

(d) Prove that $\det(\psi \circ \phi) = \det(\psi)\det(\phi) = \det(\phi \circ \psi)$ for all $\psi\colon V \to V$.

2. Let V be a F-vector space of finite dimension n.

(a) Prove that the F-bilinear map $\bigwedge^i V \times \bigwedge^{n-i} V \to \bigwedge^n V$, defined on simple wedges by the "wedging" $(v_1 \wedge \cdots \wedge v_i, v_{i+1} \wedge \cdots \wedge v_n) \mapsto v_1 \wedge \cdots \wedge v_n$, induces an isomorphism $\bigwedge^i V \cong (\bigwedge^{n-i} V)^\vee \otimes_F \bigwedge^n V$. What combinatorial formula do you see by computing the dimensions of both sides?

(b) Let $f : V \to F$ be a nonzero linear map and define $d_f^i : \bigwedge^i V \to \bigwedge^{i-1} V$ to be the F-linear map uniquely determined on simple wedges by

$$d_f^i(v_1 \wedge \cdots \wedge v_i) = \sum_{j=1}^{i} (-1)^j f(v_j)\, v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_i,$$

where $v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_i$ means the wedge monomial obtained by omitting the jth term. Prove that the sequence of maps

$$0 \to \textstyle\bigwedge^n V \xrightarrow{d_f^n} \bigwedge^{n-1} V \xrightarrow{d_f^{n-1}} \cdots \xrightarrow{d_f^3} \bigwedge^2 V \xrightarrow{d_f^2} V \xrightarrow{f} F \to 0$$

is an exact complex. This is called a *Koszul complex*. What combinatorial formula do you see by computing the Euler characteristic of this complex? **Hint.** If $\iota : W \hookrightarrow V$ is the kernel of f, then realize that the complex breaks into short exact sequences

$$0 \to \textstyle\bigwedge^i W \xrightarrow{\wedge^i \iota} \bigwedge^i V \xrightarrow{d_f^i} \bigwedge^{i-1} W \to 0$$

By the way, what combinatorial formula do you see by applying rank-nullity to these short exact sequences?

3. Recall that if A, B are F-algebras (with 1) then there is a unique structure of F-algebra on $A \otimes_F B$ with the property that

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$$

for all $a, a' \in A$ and $b, b' \in B$. Furthermore, if $F \subset K$ is a field extension, recall the extension of scalars $A_K = A \otimes_F K$, which has a unique structure of K-algebra with the property that

$$\lambda \cdot (a \otimes \alpha) = a \otimes \lambda\alpha$$

for all $a \in A$ and $\lambda, \alpha \in K$. For a K-algebra B, recall the restriction of scalars ${}_F B$, which is a F-algebra by restricting the scalar multiplication from K to F.

(a) For F-algebras A and B, verify (whenever it makes sense) that

$$\dim_F(A \otimes_F B) = \dim_F A \cdot \dim_F B$$

(b) For a F-algebra A and a K-algebra B, verify (whenever they make sense) that

$$\dim_K(A_K) = \dim_F A, \qquad \dim_F({}_F B) = \dim_F K \cdot \dim_K B$$

(c) Consider the $\mathbb{R}$-algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}$ as $\mathbb{R}$-algebras. **Hint.** Define the map $\Phi\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $(z, w) \mapsto (zw, z\overline{w})$ and use it to get an $\mathbb{R}$-algebra isomorphism $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \times \mathbb{C}$.

(d) Let $\sigma : K \to K$ be a F-algebra automorphism and W a K-vector space. Define a new K-vector space ${}^{\sigma}W$ on the same underlying abelian group W with scalar multiplication $\lambda \cdot w = \sigma(\lambda)w$ for all $\lambda \in K$ and $w \in W$. This is called the σ-twist of W. Prove that $\dim_K({}^{\sigma}W) = \dim_K W$. Prove that to give a K-linear map $\phi : W \to {}^{\sigma}W'$ is the same as to give a F-linear map $\phi : {}_FW \to {}_FW'$ such that $\phi(\lambda w) = \sigma(\lambda)\phi(w)$ for all $\lambda \in K$ and $w \in W$. Such ϕ are often called σ-semilinear.

(e) When $F = \mathbb{R}$, $K = \mathbb{C}$, and σ is complex conjugation, ${}^{\sigma}W$ is often denoted by $\overline{W}$. Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \overline{\mathbb{C}}$ as $\mathbb{C}$-algebras.

4. Let R be a commutative ring. Let J be an ideal of R.

(a) For M a R-module, let

$$JM = \left\{ \sum_{i=1}^{n} a_i x_i : a_i \in J, x_i \in M \right\}.$$

Show that JM is an R-submodule of M.

(b) If $\phi\colon M \to N$ is an isomorphism of R-modules, show that $\phi|_{JM}$ induces an isomorphism $JM \simeq JN$.

(c) Let $\{M_i\}_{i\in I}$ be R-modules and let $N_i \subset M_i$ be R-submodules for all i. Prove that

$$\left(\bigoplus_{i\in I} M_i \right) \Big/ \left(\bigoplus_{i\in I} N_i \right) \simeq \bigoplus_{i\in I} M_i/N_i.$$

(d) If $M \simeq \bigoplus_{i\in I} R$ is a free R-module, show that

$$M/JM \simeq \bigoplus_{i\in I} R/J$$

as R-modules.

(e) Suppose that R is not the zero ring. Prove that two free R-modules are isomorphic if and only if they have R-bases of the same cardinality, in particular $R^n \simeq R^m$ if and only if $n = m$. **Hint.** Apply (d) with J a maximal ideal of R.

5. Let R be an integral domain and let M be a finitely generated R-module. The *rank* of M is the maximal number of R-linearly independent elements of M.

We already defined the rank of a free R-module with basis β to be $\#\beta$, and in the previous exercise we showed it is well-defined. But if we are going to use the same word *rank*, we should show that the two notions concide when M is free (over a domain)! (If this is confusing to you, use the term *free rank* for free modules while you work on this exercise, then afterwards you can drop the "free".)

(a) Suppose that M has rank n and that $x_1, \ldots, x_n$ is any maximal set of R-linearly independent elements of M. Let $N = Rx_1 + \cdots + Rx_n$ be the R-submodule generated by $x_1, \ldots, x_n$. Prove that N is isomorphic to R^n and that the quotient M/N is a torsion R-module. **Hint.** Show that the map $R^n \to N$ which sends the ith standard basis vector to x_i is an isomorphism of R-modules.

(b) Prove conversely that if M contains a submodule N that is free of rank n (i.e., $N \simeq R^n$) such that the quotient M/N is a torsion R-module then M has rank n. **Hint.** Let $y_1, \ldots, y_{n+1}$ be any $n+1$ elements of M. Use the fact that M/N is torsion to write $r_i y_i$ as a linear combination of a basis for N for some nonzero elements r_i of R. Show that the $r_i y_i$, and hence also the y_i, are linearly dependent, working over the field of fractions.

(c) Conclude that $M = R^n$ has rank n, i.e., that the maximum number of R-linear independent elements in R^n is n (the same as the cardinality of the standard basis).

(d) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated R-modules. Prove that the rank of M is the sum of the ranks of M' and M''.