Chapter 6.1 # 2, 34, 40

2) Note that on the interval $[0, 6]$

$$2x \geq x^2 - 4x$$

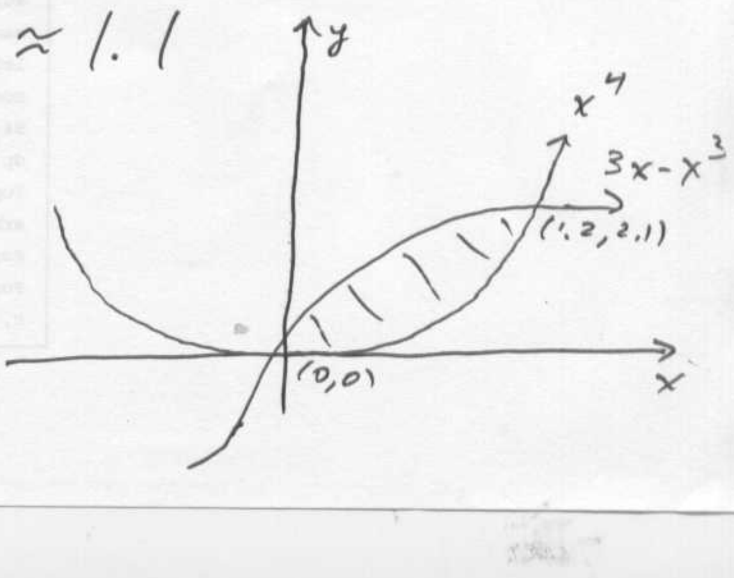
So the area bounded between the graphs of $y = 2x$ and $y = x^2 - 4x$ is given by

$$A = \int_0^6 (2x - (x^2 - 4x))\,dx = \int_0^6 (6x - x^2)\,dx$$

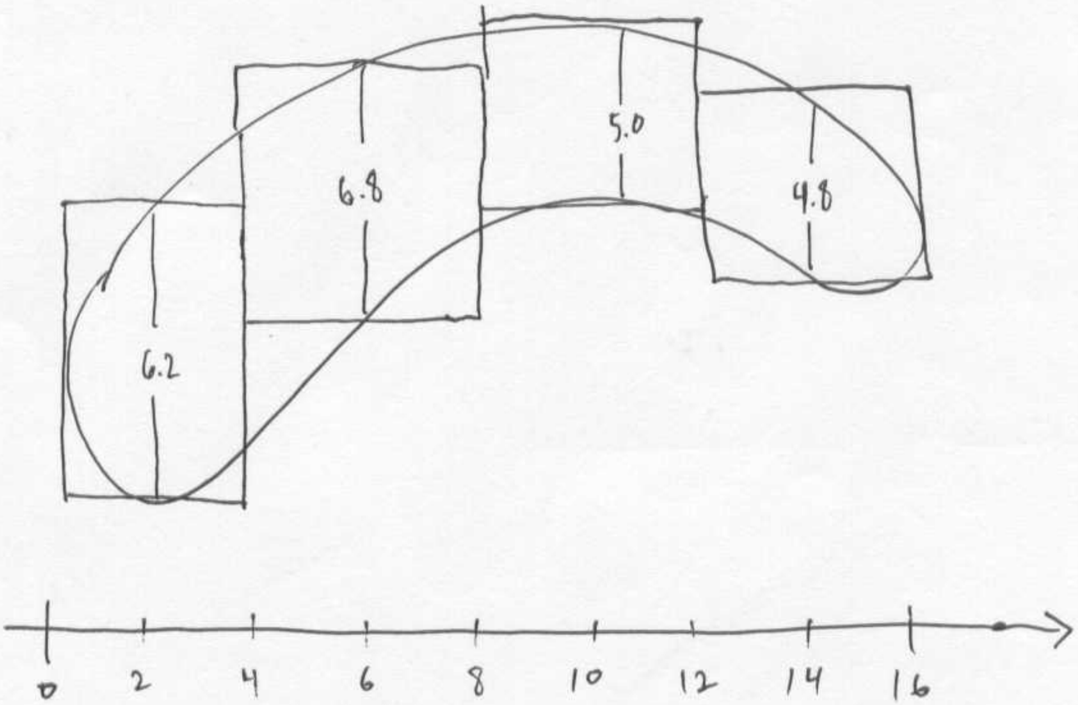
$$= 3x^2 - \tfrac{1}{3}x^3 \Big|_0^6 = 3 \cdot 6^2 - \tfrac{1}{3} \cdot 6^3 - 0 = \boxed{36}$$

34) Graphing $y = x^4$ and $y = 3x - x^3$ on Maple, I can see that the approximate points of intersection of the two graphs are $(0, 0)$ and $(1.2, 2.1)$ and that on the interval $[0, 1.2]$, $3x - x^3 \geq x^4$. Thus the approximate area between the graphs is given by

$$A \approx \int_0^{1.2} (3x - x^3 - x^4)\,dx \approx 1.1$$



40) The midpoint rule says we should approximate the kidney-shaped swimming pool by rectangles with heights given by the midpoints of some subdivision of the interval.

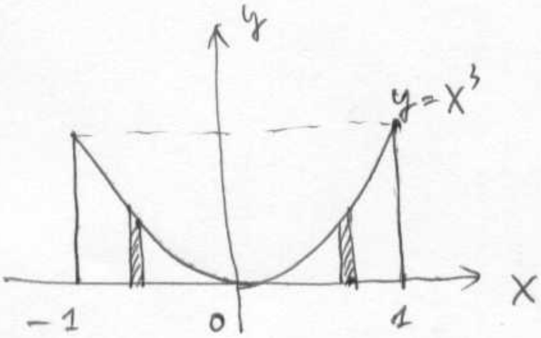


Thus the approximation is

$$4 \cdot 6.2 + 4 \cdot 6.8 + 4 \cdot 5.0 + 4 \cdot 4.8 = \boxed{91.2}$$

Core Problems 6.2

#20



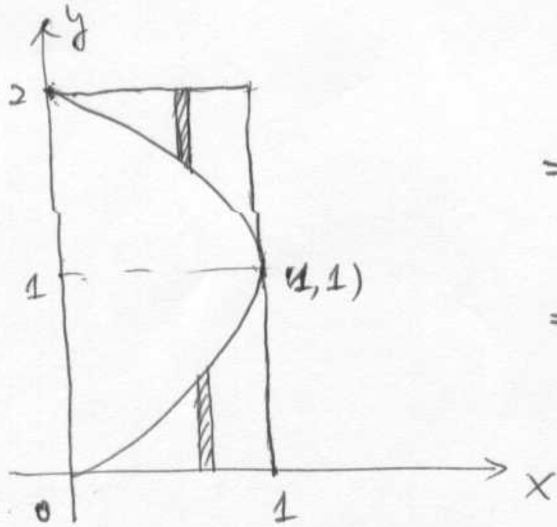
$$\int_0^1 2\pi x\, x^3\, dx$$

$$= 2\pi \int_0^1 x^4 dx$$

$$= \frac{2}{5}\pi x^5 \Big|_0^1$$

$$= \frac{2}{5}\pi$$

#22



$$\int_0^1 \pi\left(1^2 - (1-x^3)^2\right) dx$$

$$= \pi \int_0^1 (2x^3 - x^6)\, dx$$

$$= \pi \cdot \left(\frac{2}{4} x^4 \Big|_0^1 - \frac{1}{7} x^7 \Big|_0^1\right)$$

$$= \frac{5}{14}\pi$$

#32. $\displaystyle\int_2^4 2\pi(10-x)\left(8x-16-(x-2)^4\right) dx$

Core Problems 6.3

1. We only set up the integral using shell method and compute the volume V

$$V = 2\pi \int_0^1 x\, f(x)\, dx = 2\pi \int_0^1 x\, x(x-1)^2\, dx$$

$$= 2\pi \int_0^1 x^2 (x-1)^2\, dx$$

$$= 2\pi \int_0^1 \left(x^4 - 2x^3 + x^2 \right) dx$$

$$= 2\pi \left[\frac{1}{5}x^5 - \frac{2}{4}x^4 + \frac{1}{3}x^3 \right] \Big|_0^1$$

$$= 2\pi \left(\frac{1}{5} - \frac{2}{4} + \frac{1}{3} \right)$$

$$= \frac{\pi}{15}$$

2.

$$V = 2\pi \int_0^{\sqrt{\pi}} x f(x)\,dx$$

$$= 2\pi \int_0^{\sqrt{\pi}} x \sin(x^2)\,dx$$

$$= \pi \int_0^{\sqrt{\pi}} \sin(x^2)(2x)\,dx$$

$$= -\pi \int_0^{\sqrt{\pi}} \left(\cos(x^2)\right)'\,dx$$

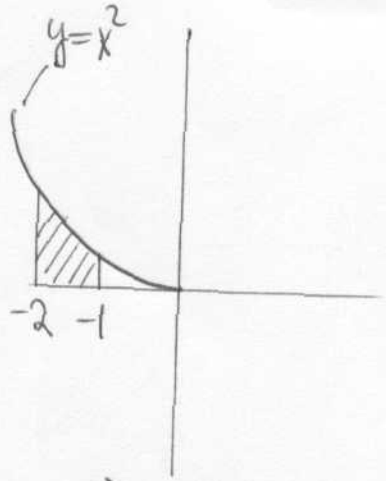
$$= -\pi \cos(x^2)\Big|_0^{\sqrt{\pi}}$$

$$= -\pi\left[\cos(\pi) - \cos(0)\right]$$

$$= 2\pi$$

16.

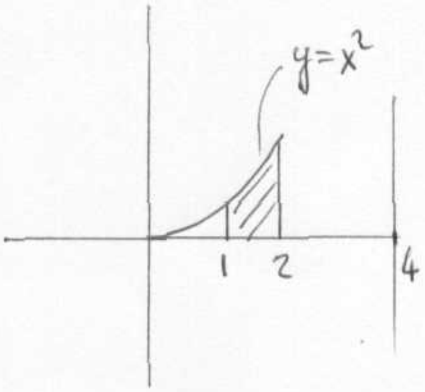
$$V = 2\pi \int_{-2}^{-1} |x| f(x)\,dx$$



$$= 2\pi \int_{-2}^{-1} (-x)x^2\,dx$$

$$= -2\pi \int_{-2}^{-1} x^3\,dx = \frac{-2\pi}{4} x^4\Big|_{-2}^{-1} = \frac{-2\pi}{4}\left(1 - (-2)^4\right) = \frac{15\pi}{2}$$

17.



$$V = 2\pi\int_1^2 (4-x)\, f(x)\,dx$$

$$= 2\pi\int_1^2 (4-x)x^2\,dx$$

$$= 2\pi\int_1^2 (4x^2-x^3)\,dx$$

$$= 2\pi\left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]\Big|_1^2$$

$$= 2\pi\left[\frac{4}{3}\cdot 8 - \frac{1}{4}\cdot 16 - \frac{4}{3} + \frac{1}{4}\right]$$

$$= \frac{67\pi}{6}$$

28. $V = 2\pi\int_2^{12} x f(x)\,dx$

$$\approx 2\pi\left[3\cdot f(3)\cdot 2 + 5f(5)\cdot 2 + 7f(7)\cdot 2 + 9f(9)\cdot 2 + 11\cdot f(11)\cdot 2\right]$$

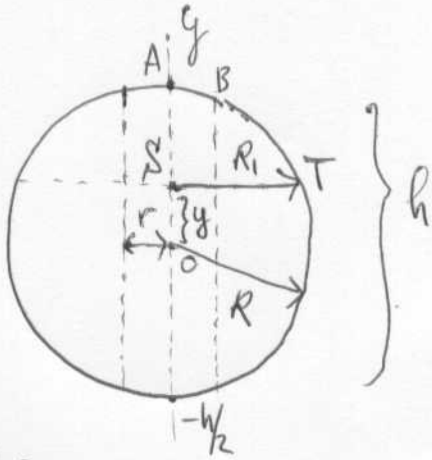
$= 332\pi$

Midpoints:

3, 5, 7, 9, 11

$\Delta x = 2$

46.



We use washer method to compute the volume.

So look at each horizontal section, say at point S, the section is a washer with outer radius R_1, inner radius r, where

$R_1 = \sqrt{R^2 - y^2}$ (look at the right triangle OST)

So

$$V = \int_{-h/2}^{h/2} \pi (R_1^2 - r^2)\, dy$$

$$= \int_{-h/2}^{h/2} \pi (R^2 - y^2 - r^2)\, dy$$

$$= \int_{-h/2}^{h/2} \pi (R^2 - r^2 - y^2)\, dy$$

$$= \int_{-h/2}^{h/2} \pi \left[\left(\frac{h}{2}\right)^2 - y^2 \right] dy \qquad \because\ R^2 - r^2 = \left(\frac{h}{2}\right)^2$$

(look at the right triangle OAB)

$$= \frac{\pi h^3}{6}$$

Section 6.4

14. A chain lying on the ground is 10 m long and it's mass is 80 kg.
How much work is required to raise one end of the chain to a height of 6 m?

First devide the chain into small parts with length Δx, measured in meters

The chain weighs 8 kg per meter, so the weight of each part is $8 \cdot \Delta x$ kg.

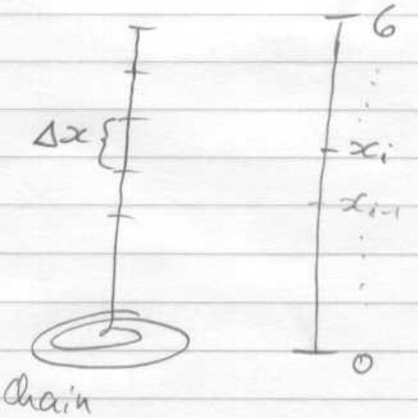
Therefore the (Gravitation) force on each part is $F = (8 \cdot \Delta x) \cdot (9.8)$
and the work required to raise this part to a height of x_i m is $W_i = F \cdot x_i = 8 \cdot \Delta x \cdot 9.8 \cdot x_i$

To get the total work W_{total} we add all these approximations and let $\Delta x \to 0$
i.e.

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} 8 \cdot 9.8 \cdot x_i \cdot \Delta x = \int_0^6 8 \cdot 9.8 \cdot x \, dx$$

$$= 8 \cdot 9.8 \cdot \tfrac{1}{2} x^2 \Big|_0^6 = 4 \cdot 9.8 \cdot 36$$

$$= \underline{\underline{1411.2 \text{ Nm}}}$$



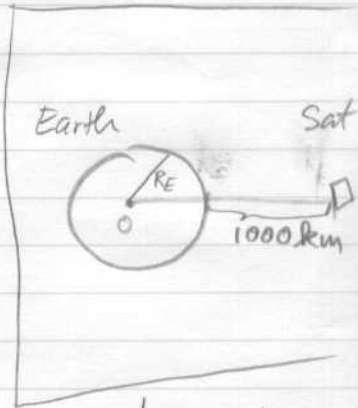
30. Use Newton's Law of Gravitation to compute the work required to launch a 1000-kg satellite vertically to an orbit 1000 km high.
You may assume that Earth's mass is 5.98×10^{24} kg and is concentrated at its center. Take the radius of Earth to be 6.37×10^6 m and $G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$

Denote the Earth's mass by $M_E = 5.98 \cdot 10^{24}$ kg and the satellite's mass by $M_s = 1000$ kg.
And let $R_E = 6.37 \cdot 10^6$ m be the Earth's radius.

Then Newton's Law of Gravitation says that the Earth and the Satellite attract each other with force
$$F = G \frac{M_E \cdot M_s}{r^2},$$
where r is the distance between both.

Now the work required to launch the satellite from the surface of the earth at $R_E = 6.37 \cdot 10^6$ m to an orbit 1000 km $= 10^6$ m high, i.e. at $R_E + 10^6$ m is

$$W = \int_{6.37 \cdot 10^6}^{6.37 \cdot 10^6 + 10^6} G \cdot \frac{M_E \cdot M_s}{x^2} \, dx$$



$$= G \cdot M_E \cdot M_s \left. -\frac{1}{x} \right|_{6.37 \cdot 10^6}^{7.37 \cdot 10^6}$$

$$= 6.67 \cdot 10^{-11} \cdot 5.98 \cdot 10^{24} \cdot 10^3 \cdot \left(-\frac{1}{7.37 \cdot 10^6} + \frac{1}{6.37 \cdot 10^6} \right)$$

$$= 39.8866 \cdot 10^{16} \cdot 10^{-6} (-0.135 + 0.156)$$

$$\approx \underline{0.837 \cdot 10^{10} \text{ Nm}}$$

6.5 AVERAGE VALUE OF A FUNCTION

(2) $f(x) = x^2 - x$, find the average value of $f(x)$ on the interval $[0, 2]$.

ANSWER

$$\text{AVERAGE VALUE} = \frac{1}{2-0} \int_0^2 (x - x^2)\,dx = \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^3}{3} \Big]_0^2 \right)$$

$$= \frac{1}{2}\left(\frac{4}{2} - \frac{8}{3} \right) = -\frac{1}{3}$$