6.5 AVERAGE VALUE OF A FUNCTION

(2) $f(x) = x^2 - x$, find the average value of $f(x)$ on the interval $[0, 2]$.

ANSWER

$$\text{AVERAGE VALUE} = \frac{1}{2-0}\int_0^2 (x - x^2)\,dx = \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^3}{3} \Big]_0^2 \right)$$

$$= \frac{1}{2}\left(\frac{4}{2} - \frac{8}{3}\right) = -\frac{1}{3}$$

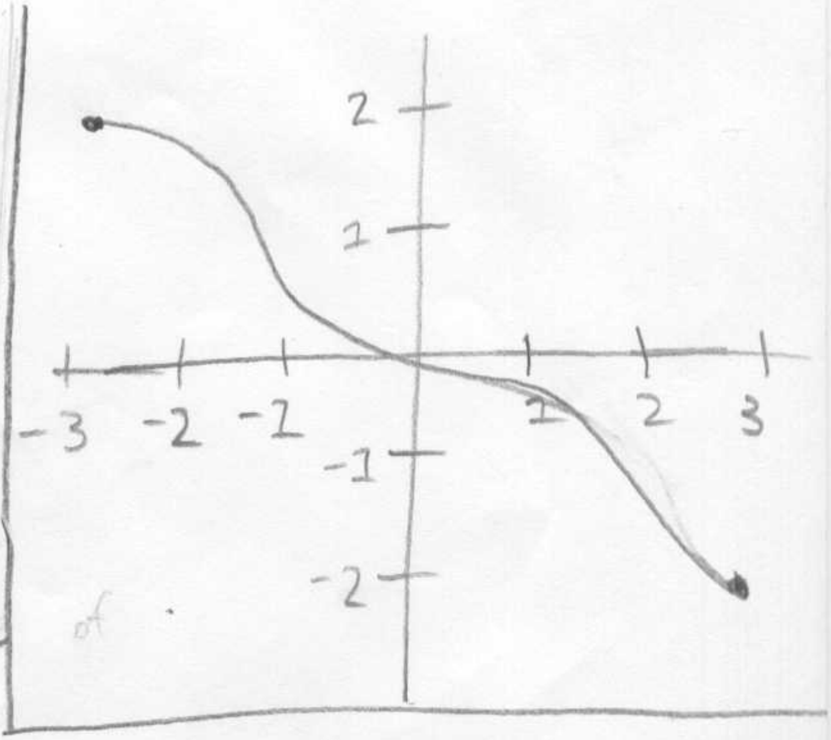
7.1 Inverse Functions

(22) The graph of f is given

a) Why is f one-to-one?

ANSWER: No horizontal line intersects the graph more than once, therefore f is one-to-ome by the horizontal line test.

Note: The graph should not be flat near zero.



b) State the Domain & Range of f^{-1}.

ANSWER

Domain of f^{-1} = Range of $f = [-2, 2]$

Range of f^{-1} = Domain of $f = [-3, 3]$

22 (continued)

c) Estimate the value of $f^{-1}(1)$.

ANSWER: $f^{-1}(1) \approx -2.2$

26 $f(x) = \dfrac{4x-1}{2x+3}$, find a formula for f^{-1}.

ANSWER:

we set $\dfrac{4x-1}{2x+3} = y$ and solve for x.

we have $4x-1 = y(2x+3)$

$$\Rightarrow 4x-1 = 2xy+3y$$

$$\Rightarrow 4x-2xy = 3y+1$$

$$\Rightarrow x(4-2y) = 3y+1$$

$$\Rightarrow x = \frac{3y+1}{4-2y}$$

$$\Rightarrow f^{-1}(y) = \frac{3y+1}{4-2y}.$$

(36) $f(x) = \sqrt{x-2}, \quad a = 2$

a) Show that f is one-to-one.

ANSWER: First note that the domain of $f(x)$ is $x \geq 2$. We now calculate the inverse function.

$$y = \sqrt{x-2}$$

$$\Rightarrow y^2 = x-2$$

$$\Rightarrow y^2+2 = x$$

Therefore $f^{-1}(y) = y^2+2. \ (= g(y))$

b) Use Theorem 7 to find $g'(a)$, where $g = f^{-1}$.

First note $f'(x) = \frac{1}{2}(x-2)^{-1/2}$.

Now $g(a) = g(2) = (2)^2+2 = 6$

So by Thm 7, $g'(a) = \dfrac{1}{f'(g(a))} = \dfrac{1}{f'(6)}$

$$= \frac{1}{\frac{1}{2}(6-2)^{-1/2}} = \frac{1}{\frac{1}{2\sqrt{4}}} = 2\sqrt{4} = 2 \cdot 2 = 4.$$

c) Calculate $g(x)$ & state the domain & range of g.

ANSWER We calculated this in part a)

(36) c) (Continued)

Range of g = Domain of f = $[2, \infty)$

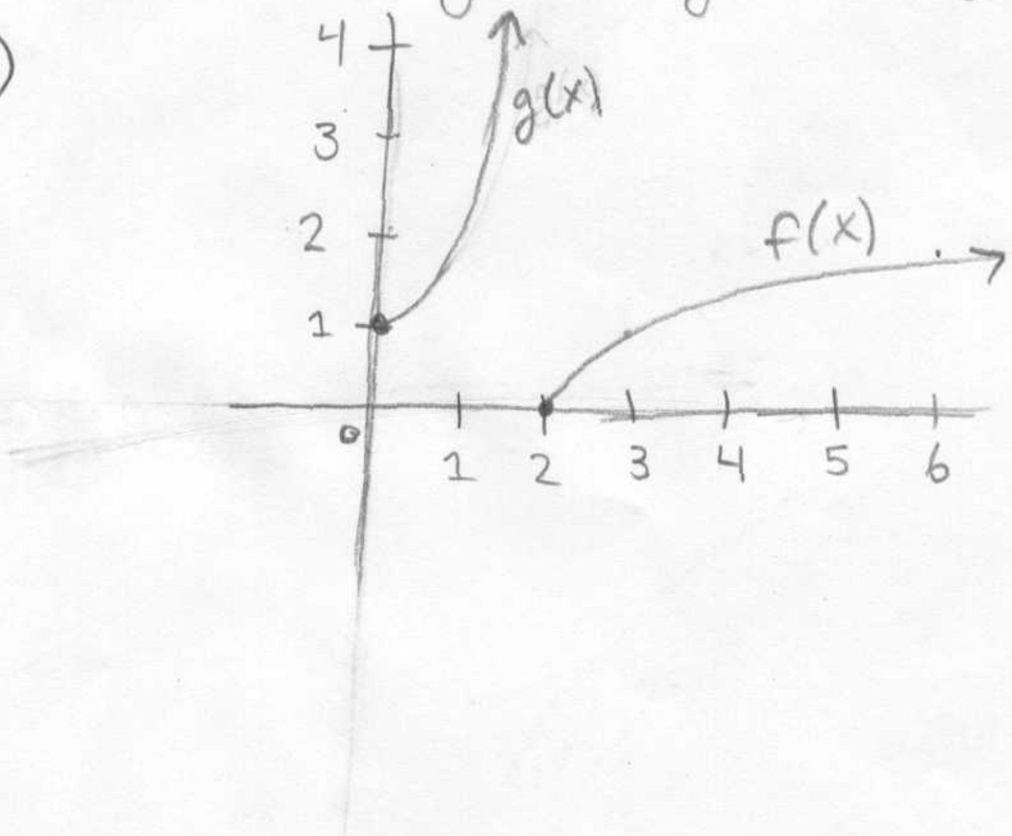
Domain of g = Range of f = $[0, \infty)$

d) Calculate $g'(a)$ directly.

$$g(y) = y^2 + 2 \implies g'(y) = 2y$$

Therefore $g'(a) = g'(2) = 2 \cdot 2 = 4$

e)



7.2 Exponential Functions & Their Derivatives

(16) Find the domain of each function

a) $g(t) = \sin(e^{-t})$

ANSWER $(-\infty, \infty) = \mathbb{R}$.

b) $g(t) = \sqrt{1 - 2^t}$

ANSWER

We require that $1 - 2^t \geq 0$

$\Longleftrightarrow \quad 1 \geq 2^t$

$\Longleftrightarrow \quad 0 \geq t$

So the domain is $(-\infty, 0]$

(30) Differentiate $f(x) = \dfrac{e^x}{1+x}$

ANSWER

We use the quotient rule to get

$$f'(x) = \frac{(1+x)e^x - e^x}{1+x} = \frac{xe^x}{1+x}$$

(38) Differentiate $f(x) = \cos(e^{\pi x})$

ANSWER

We use the chain rule to get

$$f'(x) = -\sin(e^{\pi x}) \cdot \pi e^{\pi x}$$

(44) Find an equation of the tangent line to the curve $y = e^x/x$, at the point $(1, e)$.

ANSWER We calculate the derivative of $y=f(x)$ to get the slope so.

$$f'(x) = \frac{xe^x - e^x}{x^2} \quad \text{by the quotient rule}$$

now $$f'(1) = \frac{1 \cdot e - e}{1^2} = 0 \quad \text{is the slope}$$

So $y = x + b$ is the equation of the tangent line, and we plug in $(1, e)$ to compute b.

So $e = 1 + b \Rightarrow b = e - 1$

Therefore $$y = x + e - 1.$$

(72) Evaluate $\int_0^1 x e^{-x^2} dx$.

ANSWER We use u substitution with $u = -x^2$
$du = -2x\,dx$

So that $$\int_{x=0}^{x=1} x e^{-x^2} dx = \int_{u=0}^{u=-1} x e^u \frac{du}{-2x} = \int_0^{-1} \frac{e^u}{-2} du$$

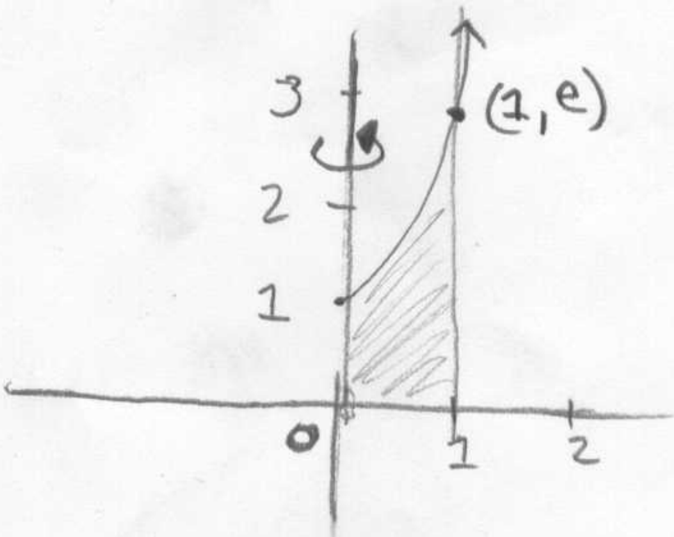
72 (continued)

$$\int_0^{-1} \frac{e^u}{-2}\,du = \int_0^1 \frac{e^u}{2}\,du = \frac{e^u}{2}\Big]_0^1 = \frac{e}{2} - \frac{1}{2}$$

82 Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curves $y = e^x$, $y=0$, $x=0$ & $x=1$.

ANSWER

PICTURE :



USING THE SHELL METHOD WE HAVE.

$$\text{Volume} = \int_0^1 2\pi x e^x\,dx$$

we use integration by parts with $u = x$, $dv = e^x dx$

So $\frac{dv}{dx} = e^x \Rightarrow v = e^x$, $du = dx$

So we have $2\pi\left(xe^x\Big]_0^1 - \int_0^1 e^x\,dx\right) = 2\pi\left(e - \int_0^1 e^x\,dx\right)$

$$= 2\pi\left(e - e^x\Big]_0^1\right) = 2\pi(e - (e-1)) = 2\pi$$

7.3 Logarithmic Functions

(2)

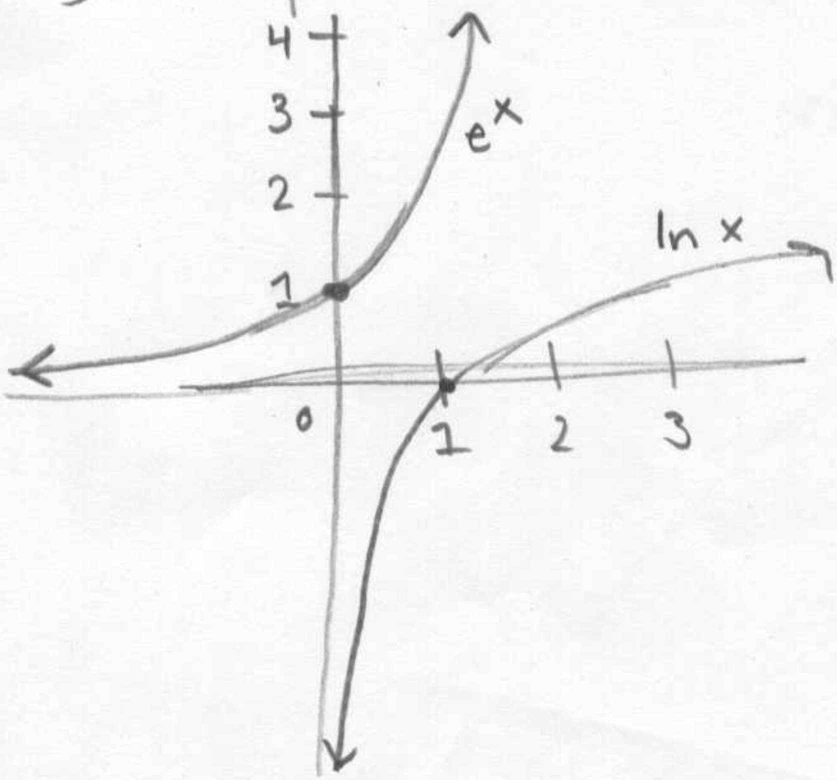
a) What is the natural logarithm?

ANSWER: The natural logarithm, denoted by $\ln x$, in the inverse function of e^x.

b) What is the common logarithm?

ANSWER: The common logarithm, denoted by $\log_a x$, is the inverse function of a^x.

c) Graph $\ln x$ & e^x.



(12) Expand $\ln \frac{3x^2}{(x+1)^5}$

ANSWER

$$\ln \frac{3x^2}{(x+1)^5} = \ln 3x^2 - \ln (x+1)^5$$

$$= \ln 3x^2 - 5\ln(x+1).$$

(18) Express $\ln x + a\ln y - b\ln z$ as a single logarithm.

ANSWER

$$\ln x + a\ln y - b\ln z = \ln x + \ln y^a - \ln z^b$$

$$= \ln \frac{xy^a}{z^b}.$$

(24) GRAPH $y = \log_2(x-3)$

Just use a calculator to check your answer

(56) Find the limit

$$\lim_{x\to\infty} \left[\ln(2+x) - \ln(1+x)\right]$$

ANSWER

$$= \lim_{x\to\infty} \left[\ln\left(\frac{2+x}{1+x}\right)\right]$$

$$= \ln 1 = 0.$$

(60) Let $f(x) = \ln(2 + \ln x)$

a) Find the domain of f.

ANSWER

We require that $x > 0$ since domain of $\ln x = (0, \infty)$

& that $2 + \ln x > 0$ for the same reason.

So $2 + \ln x > 0 \Rightarrow \ln x > -2$

$\Rightarrow x > e^{-2} = \frac{1}{e^2}$

This condition is stronger than $x > 0$

So we have that

domain of $\ln(2 + \ln x) = \left(\frac{1}{e^2}, \infty\right)$

b) Find f^{-1} & the domain of f^{-1}.

ANSWER set $y = \ln(2 + \ln x)$ and solve for x,

so $e^y = 2 + \ln x \Rightarrow \ln x = e^y - 2$

$\Rightarrow x = e^{(e^y - 2)}$

Therefore $f^{-1}(y) = e^{(e^y - 2)}$

Its domain is $(-\infty, \infty) = \mathbb{R}$.

(66) Find the inverse function of

$$y = \frac{1+e^x}{1-e^x}.$$

ANSWER We solve for x.

we have
$$y = \frac{1+e^x}{1-e^x}$$

$$\Rightarrow \quad y(1-e^x) = 1+e^x$$

$$\Rightarrow y - ye^x = 1+e^x$$

$$\Rightarrow \quad y-1 = ye^x + e^x$$

$$\Rightarrow \quad y-1 = (y+1)e^x$$

$$\Rightarrow \quad \frac{y-1}{y+1} = e^x$$

$$\Rightarrow \quad x = \ln\left(\frac{y-1}{y+1}\right).$$

This is the inverse function.