Math 104 Section 8.4

Even Core Problems

4a) First we use synthetic division:

$$
\begin{array}{r}
x - 4 \\
x^2+4x+3 \overline{\big)\, x^3 } \\
-\ \underline{x^3 + 4x^2 + 3x } \\
-4x^2 + 3x \\
-\ \underline{-4x^2 - 16x - 12} \\
19x + 12
\end{array}
$$

so that

$$\frac{x^3}{x^2+4x+3} = x - 4 + \frac{19x+12}{x^2+4x+3}$$

$$= x - 4 + \frac{19x+12}{(x+3)(x+1)}$$

$$= x - 4 + \frac{A}{x+3} + \frac{B}{x+1}$$

with A and B to be determined.

b)

$$\frac{2x+1}{(x+1)^3(x^2+4)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

since x^2+4 is irreducible by noting that its discriminant $0^2 - 4\cdot 1\cdot 4 = -16 < 0$.

10. First we write the partial fractions decomposition:

$$\frac{1}{(t+4)(t-1)} = \frac{A}{t+4} + \frac{B}{t-1}$$

$$\Leftrightarrow \quad 1 = A(t-1) + B(t+4)$$

$$0\cdot t + 1 = (A+B)t + (-A+4B)$$

$$\Leftrightarrow \begin{cases} A+B=0 \\ -A+4B=1 \end{cases} \Leftrightarrow \begin{cases} A=-B \\ 5B=1 \end{cases}$$ by substituting

$$\Leftrightarrow \begin{cases} A = -1/5 \\ B = 1/5 \end{cases}$$

Now we proceed:

$$\int \frac{1}{(t+4)(t-1)}\,dt = \int \left(\frac{-1/5}{t+4} + \frac{1/5}{t-1} \right) dt$$

$$= -\frac{1}{5}\int \frac{1}{t+4}\,dt + \frac{1}{5}\int \frac{1}{t-1}\,dt$$

$$= -\frac{1}{5}\ln|t+4| + \frac{1}{5}\ln|t-1| + C$$

$$= \ln\left|\frac{t-1}{t+4}\right|^{1/5} + C .$$

12. First we decompose into partial fractions:

$$\frac{x-1}{x^2+3x+2} = \frac{x-1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

$$\Leftrightarrow \quad x-1 = A(x+1) + B(x+2)$$

$$= (A+B)x + (A+2B)$$

$$\Leftrightarrow \begin{cases} A+B=1 \\ A+2B=-1 \end{cases} \Leftrightarrow \begin{cases} A+B=1 \\ B=-2 \end{cases}$$ by subtracting the first equation from the second.

$$\Leftrightarrow \begin{cases} A=3 \\ B=-2 \end{cases}$$

And now we proceed:

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1 \left(\frac{3}{x+2} + \frac{-2}{x+1}\right) dx$$

$$= 3\int_0^1 \frac{1}{x+2}\,dx - 2\int_0^1 \frac{1}{x+1}\,dx = 3\ln|x+2|\Big|_0^1 - 2\ln|x+1|\Big|_0^1$$

$$= 3(\ln 3 - \ln 2) - 2(\ln 2 - \ln 1)$$

$$= 3\ln\frac{3}{2} - 2\ln 2 = \ln\left(\frac{3}{2}\right)^3 - \ln 2^2$$

$$= \ln\frac{27}{32}.$$

$$= -5\ln 2 + 3\ln 3.$$

Math 104 Section 8.5
Even Core Problems

12. think what's ugly: the $\cos(\cos x)$.

So we'll try $u = \cos x$

$du = -\sin x$

then

$$\int \sin x \cdot \cos(\cos x)\,dx = -\int \cos(u)\,du = -\sin u$$

then substituting back

$$\int \sin x \cdot \cos(\cos x)\,dx = -\sin(\cos x) + C$$

60. Think what's ugly: the $\sqrt[3]{x}$.

So try $u = \sqrt[3]{x}$ then $u^3 = x$ and $3u^2\,du = dx$

then

$$\int \frac{1}{x + \sqrt[3]{x}}\,dx = \int \frac{3u^2}{u^3+u}\,du = 3\int \frac{u^2}{u(u^2+1)}\,du$$

$$= 3\int \frac{u}{u^2+1}\,du$$, now let $v = u^2+1$

$dv = 2u\,du$

so that

$$\int \frac{1}{x+\sqrt[3]{x}}\,dx = 3\int \frac{u}{u^2+1}\,du = \frac{3}{2}\int \frac{dv}{v} = \frac{3}{2}\ln|v|$$

$$= \frac{3}{2}\ln|u^2+1| = \frac{3}{2}\ln|x^{2/3}+1| + C$$

74. Try this method: let $u = e^x$.

Then $\ln u = x$ so $\frac{1}{u}\,du = dx$ and thus

$$\int \frac{1}{e^x - e^{-x}}\,dx = \int \frac{1}{u - u^{-1}} \cdot \frac{1}{u}\,du = \int \frac{1}{u^2 - 1}\,dx = \int \frac{1}{(u+1)(u-1)}\,du$$

So we decompose into partial fractions

$$\frac{1}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1} \iff 1 = A(u-1) + B(u+1)$$
$$= (A+B)u + (-A+B)$$

$$\iff \begin{Bmatrix} A + B = 0 \\ -A + B = 1 \end{Bmatrix} \iff \begin{Bmatrix} -A = B \\ 2B = 1 \end{Bmatrix} \iff \begin{Bmatrix} A = -1/2 \\ B = 1/2 \end{Bmatrix}$$

So that finally

$$\int \frac{1}{e^x - e^{-x}}\,dx = \int \frac{1}{(u+1)(u-1)}\,du = \int \left(\frac{-1/2}{u+1} + \frac{1/2}{u-1} \right) du$$

$$= -\frac{1}{2} \int \frac{1}{u+1}\,du + \frac{1}{2} \int \frac{1}{u-1}\,du = -\frac{1}{2} \ln|u+1| + \frac{1}{2} \ln|u-1|$$

$$= \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| = \frac{1}{2} \ln \left| \frac{e^x - 1}{e^x + 1} \right| = \frac{1}{2} \ln \left| \frac{e^{-x/2}}{e^{-x/2}} \frac{(e^x - 1)}{(e^x + 1)} \right|$$

$$= \frac{1}{2} \ln \left| \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} \right| = \frac{1}{2} \ln \left| \tanh \frac{x}{2} \right| .$$

76 Use integration by parts:

$$\int (x^2-bx)\sin 2x\,dx = -\frac{1}{2}(x^2-bx)\cos 2x + \frac{1}{2}\int (2x-b)\cos 2x\,dx$$

$u = x^2 - bx \qquad dv = \sin 2x\,dx$

$du = (2x-b)\,dx \qquad v = -\frac{1}{2}\cos 2x$

and now use integration by parts again:

$$\int (2x-b)\cos 2x\,dx = \frac{1}{2}(2x-b)\sin 2x - \frac{1}{2}\cdot 2\int \sin 2x\,dx$$

$u = 2x - b \qquad dv = \cos 2x\,dx$

$du = 2 \qquad v = \frac{1}{2}\sin 2x$

$$= \frac{1}{2}(2x-b)\sin 2x + \frac{1}{2}\cos 2x$$

So combing everything we get

$$\int (x^2-bx)\sin 2x\,dx$$

$$= -\frac{1}{2}(x^2-bx)\cos 2x + \frac{1}{4}(2x-b)\sin 2x + \frac{1}{4}\cos 2x$$

$$= -\frac{1}{2}\left(x^2-bx-\frac{1}{2}\right)\cos 2x + \frac{1}{4}(2x-b)\sin 2x.$$

Math 104 Section 8.6

Even Core Problems

6 To pretty it up and make it look more like reference page 7's formula # 45, let $u = 2x$ with $du = 2\,dx$, then

$$\int \frac{1}{x^2\sqrt{4x^2-7}}\,dx = \int \frac{1/2}{(u/2)^2\sqrt{u^2-7}}\,du = 2\int \frac{du}{u^2\sqrt{u^2-7}}$$

then use formula # 45 with $a = \sqrt{7}$ to get
$a^2 = 7$

$$\int \frac{1}{x^2\sqrt{4x^2-7}}\,dx = 2 \cdot \frac{\sqrt{u^2-7}}{7u}$$

$$= 2 \cdot \frac{\sqrt{4x^2-7}}{14x} = \frac{\sqrt{4x^2-7}}{7x}$$

and so finally

$$\int_2^3 \frac{dx}{x^2\sqrt{4x^2-7}} = \frac{\sqrt{4x^2-7}}{7\cdot x}\Bigg|_2^3 = \frac{\sqrt{4\cdot 9-7}}{21} - \frac{\sqrt{4\cdot 4-7}}{14}$$

$$= \frac{\sqrt{29}}{21} - \frac{3}{14} = \frac{1}{21}(\sqrt{29} - 9)$$

20 Let $u = e^x$, $\ln u = x$, $dx = \frac{1}{u}\,du$, then

$$\int \frac{dx}{e^x(1+2e^x)} = \int \frac{du}{u^2(1+2u)} = -\frac{1}{u} + 2\ln\left|\frac{1+2u}{u}\right| + C$$

by formula #50 on pg. 8 with $a = 1, b = 2$

$$= -\frac{1}{e^x} + 2\ln\left|\frac{1+2e^x}{e^x}\right| + C$$