UNIVERSITY OF PENNSYLVANIA DEPARTMENT OF MATHEMATICS
Math 104 Fall Semester 2005
Prof. Cohen, T.A. Asher Auel

Test #3 Answer Key

1) Find the surface area generated by rotating, about the x-axis, the curve

$$x = 3t - t^3, \quad y = 3t^2, \quad 0 \le t \le 1$$

Answer: First we compute

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (3-3t^2)^2 + (6t)^2 = 9((1-t^2) + (2t)^2) \\ &= 9(1 - 2t^2 + t^4 + 4t^2) = 9(1 + 2t^2 + t^4) = 9(1+t^2)^2 \end{aligned}$$

and so

$$\begin{aligned} S &= \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= 6\pi \int_0^1 t^2 \sqrt{9(1+t^2)^2} dt = 18\pi \int_0^1 t^2(1+t^2) dt \\ &= 18\pi \left(\frac{t^3}{3} + \frac{t^5}{5}\right)\bigg|_0^1 = 18\pi\left(\frac{1}{3} + \frac{1}{5}\right) = \frac{48\pi}{5} \end{aligned}$$

2) Does $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\pi/n)}{\pi/n}$ converge?

Answer: Since I don't immediately understand the summand, lets try the test for divergence first:

$$\lim_{n\to\infty} \frac{\sin(\pi/n)}{\pi/n} = \lim_{n\to\infty} \frac{\cos(\pi/n)(-\pi/n^2)}{-\pi/n^2} = \lim_{n\to\infty} \cos(\pi/n) = \cos(0) = 1 \neq 0,$$

where the first equality is by l'Hôpital's rule, since the quotient is of the form "$\frac{0}{0}$". So by the test for divergence (since the limit of the sequence isn't 0), the series diverges.

3) Does $\displaystyle\int_1^{\infty} \frac{\ln(x)}{x}\, dx$ converge? If so, what is the value of the integral?

Answer: We just compute the improper integral to see what's going on. First, for the anti-derivative, we use integration by parts:

$$\int \frac{1}{x^2} \ln(x)\, dx = -\frac{1}{x}\ln(x) - \int \left(-\frac{1}{x}\right)\frac{1}{x}\, dx = -\frac{1}{x}\ln(x) - \frac{1}{x} = -\frac{1}{x}(\ln(x) + 1)$$

Now,

$$\int_1^{\infty} \frac{\ln(x)}{x}\, dx = \lim_{t\to\infty} \int_1^t \frac{\ln(x)}{x}\, dx = \lim_{t\to\infty} -\frac{\ln(x)+1}{x}\bigg|_1^t = \lim_{t\to\infty} -\frac{\ln(t)+1}{t} + 1 = 1,$$

since

$$\lim_{t\to\infty} -\frac{\ln(t)+1}{t} = \lim_{t\to\infty} -\frac{1/t}{1} = 0,$$

by l'Hôpital's rule since the quotient is of the form "$\frac{\infty}{\infty}$." So the integral converges and equals 1.

4(A) Does $\displaystyle\sum_{n=0}^{\infty} \frac{|\sin(n\pi/4)|}{3^n}$ converge?

Answer: The summand looks like a geometric series, and he numerator is always less than 1, so

$$\frac{|\sin(n\pi/4)|}{3^n} \leq \frac{1}{3^n} = \left(\frac{1}{3}\right)^n, \quad \text{for all } n \geq 0.$$

Since the series $\sum_{n=0}^{\infty}(1/3)^n$ converges (since it's geometric series with $r = 1/3 < 1$) then by the comparison test, our series converges.

(B) Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3}{2n^3-n}$ converge?

Answer: When n is big, the summand looks like $\dfrac{n^2}{2n^3} = \dfrac{1}{2n}$ which diverges by the p-series test. To make this rigorous, note that in fact

$$\frac{n^2+3}{2n^3-n} \geq \frac{n^2}{2n^3} = \frac{1}{2n}, \quad \text{for all } n \geq 1,$$

so that by the comparison test (our series is always bigger than a divergent series), our series diverges.

5) Find the interval of convergence of the power series $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^{n+2}x^n}{\sqrt{n-1}}$.

Answer: We use the ratio test:

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+3}x^{n+1}}{\sqrt{n}} \frac{\sqrt{n-1}}{(-2)^{n+2}x^n} \right| = \lim_{n\to\infty} |2x| \sqrt{\frac{n-1}{n}} = 2|x|,$$

so that the series converges for all $2|x| < 1$, i.e. for all $|x| < 1/2$, and diverges for all $|x| > 1/2$. So the radius of convergence of the power series is $1/2$, and also note that the power series is centered at the origin. Now to test the endpoints of the interval:

$$x = \frac{1}{2}; \quad \sum_{n=2}^{\infty} \frac{(-2)^{n+2}(1/2)^n}{\sqrt{n-1}} = 4\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n-1}},$$

is convergent by the alternating series test. Here you need to check (or just note) that the function $1/\sqrt{n-1}$ is decreasing and goes to 0 as $n \to \infty$. Also

$$x = -\frac{1}{2}; \quad \sum_{n=2}^{\infty} \frac{(-2)^{n+2}(-1/2)^n}{\sqrt{n-1}} = 4\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}},$$

diverges by the p-series test, since here $p = 1/2 \leq 1$.

Thus finally, the interval of convergence is $(\frac{1}{2}, \frac{1}{2}]$.

6 (A) Graph the polar equation $r = \dfrac{1}{2+\sin\theta}$. Label the focus, directrix, and vertices.
Answer: First we write

$$r = \frac{1}{2+\sin\theta} = \frac{\frac{1}{2}}{1+\frac{1}{2}\sin\theta} = \frac{ed}{1+e\sin\theta},$$

and so we see that this is the polar equation of a conic with eccentricity $e = 1/2$ thus an ellipse, and directrix the line $y = 1$ since we have a positive sine in the denominator. The focus F, or in this case, since we have an ellipse, one of the foci, is at the origin, and the ellipse is taller than it is wide. Note that the y-intercepts (the "top" and "bottom" vertices) are

$$(r,\theta) = (1/3, \pi/2),\ (1, 3\pi/2) \quad \text{or} \quad (x,y) = (0, 1/3),\ (0,-1).$$

Thus the center C of the ellipse is at the point $(x, y) = (0, -1/3)$, and the other focus F', being equidistant to the center with the focus at the origin, is at the point $(x, y) = (0, -2/3)$.

To find the "right" and "left" vertices, we need to find the intercepts with the center line $y = -1/3$. To this end, we note that using the other description of the ellipse, as the points in the plane whose sum of distances to the two foci is constant, the sum of the distances to the foci is $4/3$ by checking at either the top or bottom vertex. Thus we have a triangle FCv where v is either the left or right vertex, with hypotenuse $4/6$ (that's the constant distance $4/3$ divided by two) and the vertical side length $1/3$, so by the Pythagorean theorem, the horizontal side length (or the x-component of the vertex) is $\sqrt{3}/3$. So finally, the left and right vertices are

$$(x,y) = (-\sqrt{3}/3, -1/3),\ (\sqrt{3}/3, -1/3).$$

You can fill in the picture yourself.

(B) What is the eccentricity of the above curve? Which type of conic is it?
Answer: We already answered that, $e = 1/2$ and so the conic is an ellipse.

(C) Find the slope of the tangent at $\theta = \pi/4$.
Answer: First we compute

$$\frac{dr}{d\theta} = \frac{-\cos\theta}{(2+\sin\theta)^2}.$$

Now the slope of the tangent is given by dy/dx, and we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\frac{-\sin\theta\,\cos\theta}{(2+\sin\theta)^2} + \frac{\cos\theta}{2+\sin\theta}}{\frac{-\cos^2\theta}{(2+\sin\theta)^2} - \frac{\sin\theta}{2+\sin\theta}} \\ &= \frac{-\sin\theta\,\cos\theta + \cos\theta\,(2+\sin\theta)}{-\cos^2\theta - \sin\theta\,(2+\sin\theta)} = \frac{2\cos\theta}{-1-2\sin\theta}, \end{aligned}$$

where in the second line, I cleared the ugly denominators by multiplying top and bottom by $(2+\sin\theta)^2$. So evaluating at $\theta = \pi/4$ gives

$$\frac{dy}{dx} = \frac{2\frac{\sqrt{2}}{2}}{-1-2\frac{\sqrt{2}}{2}} = \frac{-\sqrt{2}}{1+\sqrt{2}} = \sqrt{2} - 2.$$