Dartmouth College Department of Mathematics
Math 125 Current Problems in Number Theory:
Galois Cohomology and Descent
Winter 2022

Group Work # 4 (Thursday, February 17)

Reading: *Gill–Szamuely* §3.2, 4.1

Group Work: To be discussed during the second half of class on Thursday, with the discussion led by a student selected ahead of time.

Notation: Let L/K be a finite Galois extension with Galois group G. Recall that the usual field-theoretic norm map $N_{L/K} : L \to K$ is given by $N_{L/K}(x) = \prod_{\sigma \in G} \sigma(x)$.

1. Let G be a finite cyclic group of order n and fix a generator σ. Let A be a G-module. Consider the maps $N : A \to A$ and $\sigma - 1 : A \to A$ defined by

$$N(x) = \sum_{i=0}^{n-1} \sigma^i(x) \qquad \text{and} \qquad (\sigma - 1)(x) = \sigma(x) - x.$$

(a) Verify that the $\mathbb{Z}[G]$-module $\mathbb{Z}$ has a free resolution

$$\cdots \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ is the usual **augmentation** map sending every group element to 1. Hint: Finally learn about homotopy retractions.

(b) Show that this resolution gives the following periodicity on the level of cohomology

$$H^0(G, A) = A^G \qquad \text{and} \qquad H^i(G, A) = \begin{cases} {}_N A/(\sigma - 1)A & \text{if } i \text{ is odd} \\ A^G/NA & \text{if } i \text{ is even} \end{cases}$$

for $i > 0$, where ${}_N A = \ker(N : A \to A)$.

(c) Give formulas for $H^i(G, A)$ when G acts trivially on A. For example, for G acting trivially on $\mathbb{Z}$ and on $\mathbb{Z}/n\mathbb{Z}$, compute $H^2(G, \mathbb{Z})$ and $H^2(G, \mathbb{Z}/n\mathbb{Z})$. What is the interpretation in terms of group extensions?

(d) Now, do the above three parts when G is infinite cyclic (e.g., $G = \mathbb{Z}$). Hint: The free resolution, as above, is quite short, hence the cohomology is no longer periodic, but vanishes in high degree!

2. Let L/K be a finite Galois extension with cyclic Galois group G.

(a) Use the cohomology of cyclic groups to show that the cohomological form of Hilbert's Theorem 90, namely $H^1(G, L^\times) = 1$, is equivalent to the classical form: that $x \in L^\times$ satisfies $N_{L/K}(x) = 1$ if and only if $x = \sigma(y)/y$ for some $y \in L^\times$.

(b) Use the cohomology of cyclic groups to prove that $H^2(G, L^\times) \cong K^\times/N_{L/K}(L^\times)$.

3. Let K be a field of characteristic 0 with algebraic (also separable) closure $\overline{K}$. Assume that the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ is cyclic of prime order p.

(a) Prove that $H^2(K, \overline{K}^\times) \cong K^\times/K^{\times p}$. Hint: Use the long exact sequence in Galois cohomology associated to the Kummer sequence, along with Hilbert's Theorem 90, and the periodicity of the cohomology of cyclic groups.

(b) Conclude that $N_{\overline{K}/K}(\overline{K}^\times) = K^{\times p}$ and hence that the only possibility is $p = 2$ and $\overline{K} = K(\sqrt{-1})$. Hint: Show that K contains a primitive pth root of unity (if not try adjoining it), hence that the cyclic extension $\overline{K}/K$ is a Kummer extension, i.e., $\overline{K} = K(\alpha)$ where $\alpha^p = y$ for some $y \in K^\times \smallsetminus K^{\times p}$, then try computing $N_{\overline{K}/K}(\alpha)$.

(c) Show that declaring the squares in $K^\times$ to be positive will equip K with the structure of an ordered field.

(d) (Artin–Schreier) Prove that if K is a field of characteristic 0 whose absolute Galois group is a nontrivial finite group, then $\overline{K} = K(\sqrt{-1})$ and K is an ordered field where the squares are positive. Hint: Take a p-Sylow subgroup of the Galois group and use the fact that p-groups are solvable, then iteratively apply the previous results.

Remark. Such fields are called **real closed**. In fact, Artin and Schreier proved that in positive characteristic, the absolute Galois group is either trivial or infinite.

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