MATH 170 IDEAS IN MATHEMATICS (SUMMER 2006)

Problem Set 2: Formal Systems and Isomorphisms

Due in class Tuesday 23rd, May 23rd

1. The formal system D_4

Definition 1. We'll define a formal system called D_4 by specifying the following symbols, axioms, and production rules (sometimes called *relations*):

Symbols:	e, s, t
Axioms:	e
Rules:	for any (nonempty) strings x and y, we have the following rules:
	I $x \mapsto xs$ and $x \mapsto xt$
	II $xe \mapsto x$ and $xey \mapsto xy$
	III $ss \mapsto e$ and $tttt \mapsto e$
	IV $st \mapsto ttts$
	and where you can perform rules III and IV anywhere in a string.

We'll call theorems of the formal system D_4 simply *elements* of D_4.

Answer the following questions about D_4:

a. Can you determine a decision procedure for a string to be an element of D_4?

b. Show that, starting from any element of D_4, and using only rules II-IV repeatedly, you can derive one **and only one** of the following elements:

$$e, et, ett, ettt, es, ets, etts, ettts,$$

and call this the *reduced form* of the element. For example the element $ests$ is not one of the reduced forms, but we can make the derivation:

$$\begin{aligned} ests &\mapsto etttss \quad \text{(IV)} \\ &\mapsto ettte \quad \text{(III)} \\ &\mapsto ettt \quad \text{(II)} \end{aligned}$$

and so the reduced form of $ests$ is $ettt$.

c. You can "multiply" strings by concatenating them. For example, et times es is just $etes$, which has reduced form ets. What about es times et? Is this multiplication *commutative*, i.e. does it depend on the order of the two elements being multiplied?

d. Fill in the following "multiplication table" for the elements in reduced form of D_4, i.e. pick an elements in reduced form from the left hand column and multiply it by an element in reduced form in the top row, then write the reduced form of the product in the corresponding box of the table. For your convenience, the example from part **c.** is already included on the table. You don't need to write out all of your derivations, perhaps just three of your favorite.

D_4	e	et	ett	$ettt$	es	ets	$etts$	$ettts$
e								
et					ets			
ett								
$ettt$								
es								
ets								
$etts$								
$ettts$								

Do you have any general observations about the above multiplication table?

2. Symmetries of a square

Consider the square with all its corners numbered:

$$\begin{array}{ccc} 1 & — & 2 \\ | & & | \\ 4 & — & 3 \end{array}$$

A *symmetry* of the numbered square is any symmetry of the square by rotation or flips, only allowing the numbers on the corners to change "along with" the symmetry. For example you can rotate clockwise by 90° or flip the square about the (upper left to lower right) diagonal:

$$\begin{array}{ccc} 1 & — & 2 \\ | & & | \\ 4 & — & 3 \end{array} \xrightarrow{\text{rotate}} \begin{array}{ccc} 4 & — & 1 \\ | & & | \\ 3 & — & 2 \end{array}, \quad \text{or} \quad \begin{array}{ccc} 1 & — & 2 \\ | & & | \\ 4 & — & 3 \end{array} \xrightarrow{\text{flip}} \begin{array}{ccc} 1 & — & 4 \\ | & & | \\ 2 & — & 3 \end{array},$$

respectively. The following is NOT a symmetry of the numbered square:

$$\begin{array}{ccc} 1 & — & 3 \\ | & & | \\ 4 & — & 2 \end{array}$$

Answer the following questions:

a. How many symmetries of the numbered square are there? Include in your count the "no nothing" symmetry:

$$\begin{array}{ccc} 1 & — & 2 \\ | & & | \\ 4 & — & 3 \end{array} \xrightarrow{\text{do nothing}} \begin{array}{ccc} 1 & — & 2 \\ | & & | \\ 4 & — & 3 \end{array}$$

Draw each one.

b. You can "multiply" symmetries by composing them. Denote by r the clockwise 90° rotation and by f the flip as shown above. Then $f \cdot r$ is the symmetry gotten by first rotating then flipping displayed above at right. Notice that some compositions yield the same symmetry:

$$fr \ : \ \begin{array}{ccc} 1 & — & 2 \\ | & & | \\ 4 & — & 3 \end{array} \xrightarrow{r} \begin{array}{ccc} 4 & — & 1 \\ | & & | \\ 3 & — & 2 \end{array} \xrightarrow{f} \begin{array}{ccc} 4 & — & 3 \\ | & & | \\ 1 & — & 2 \end{array}$$

$$rrrf \ : \ \begin{array}{ccc} 1 & — & 2 \\ | & & | \\ 4 & — & 3 \end{array} \xrightarrow{f} \begin{array}{ccc} 1 & — & 4 \\ | & & | \\ 2 & — & 3 \end{array} \xrightarrow{r} \begin{array}{ccc} 2 & — & 1 \\ | & & | \\ 3 & — & 4 \end{array} \xrightarrow{r} \begin{array}{ccc} 3 & — & 2 \\ | & & | \\ 4 & — & 1 \end{array} \xrightarrow{r} \begin{array}{ccc} 4 & — & 3 \\ | & & | \\ 1 & — & 2 \end{array},$$

and so we say that $fr = rrrf$. In this naming scheme, denote by n the "no nothing symmetry." Show that $ff = n$ and $rrrr = n$.

c. Now, write each symmetry of the square you have listed in part **a.** as a product of r's and f's. Do you see any relationship between the r's and f's and the s's and t's from the formal system D_4?

3. An isomorphism

Show that under the interpretation:

$$e \longrightarrow n$$
$$s \longrightarrow f$$
$$t \longrightarrow r$$

there's an isomorphism between the elements in reduced form of the formal system D_4 and the symmetries of the numbered square, and furthermore, that this isomorphism preserves the "multiplication" in each system.

This can also be stated in terms of multiplication tables, i.e. if you write out the multiplication table for the symmetries of the square, then matching up the elements in reduced form of D_4 and the symmetries of the square via the interpretation makes their respective multiplication tables match up.

Note: The formal system D_4 is usually called the *dihedral* 4*-group*. There are also dihedral n-groups D_n for all $n \geq 3$ and are isomorphic to the symmetries of the regular numbered n-gon.