MATH 170 IDEAS IN MATHEMATICS (SUMMER 2006)

Problem Set 6: Exploring the propositional calculus.

Due in class Tuesday, June 6th

1. Well defined strings

Definition 1 (Well Formed Strings)**.** Recall the definition of the formal system whose theorems are the *well formed strings*:

symbols There are three types of symbols:

- proposition variables: $P,\ Q,\ R,\ \ldots$
- logical symbols: $\wedge,\ \vee,\ \Rightarrow,\ \neg$
- angle brackets: $<,\ >$

axioms All proposition variables $P,\ Q,\ R,\ \ldots$ are axioms.

rules If x and y are well formed strings then so are the following:

$$\neg x, \quad < x \wedge y >, \quad < x \vee y >, \quad < x \Rightarrow y > .$$

Now answer the following questions:

a. Make a few well formed strings. Make the well formed string that "means": if either P or Q then either P or not Q.

b. What is the point of the well formed strings? Why are they special compared to just arbitrary strings?

c. Can you attach a truth table to each well formed string? Give an example of an arbitrary string for which attaching a truth table doesn't make much sense. Can you attach a truth table to an arbitrary string? Make the truth tables for your well formed strings in part **a.**

2. Derivations and the Fantasy Rule

Definition 2. Recall the definition of the formal system *Propositional Calculus (PC)*:

symbols As before, there are three types of symbols

- proposition variables: $P,\ Q,\ R,\ \ldots$
- logical symbols: $\wedge,\ \vee,\ \Rightarrow,\ \neg$
- angle brackets: $<,\ >$

axioms There are no axioms.

rules There are three types of rules:

I Rules that define the "meanings" of the logical symbols:

- $\wedge$ if x and y are theorems then so is $< x \wedge y >$ and if $< x \wedge y >$ is a theorem then so are both x and y
- $\Rightarrow$ if x and $< x \Rightarrow y >$ are theorems then so is y
- $\neg$ the string $\neg\neg$ may be removed from any theorem and may be added to any theorem provided that the resulting string is still well-formed

II Rules that govern how the logical symbols interact. If x and y are theorems, then

- de Morgan's Rule: $\neg < x \vee y >$ is interchangeable with $< \neg x \wedge \neg y >$
- Contrapositive: $< x \Rightarrow y >$ is interchangeable with $< \neg y \Rightarrow \neg x >$
- Switcheroo: $< x \Rightarrow y >$ is interchangeable with $< \neg x \vee y >$

III The fantasy rule: if from a well formed string x you can derive y in a "fantasy" then $< x \Rightarrow y >$ is a theorem. Inside a fantasy you can again use the fantasy rule (now you're in a metafantasy) and any theorem from the fantasy is still a theorem in the metafantasy.

Also, we will always make the interpretation of this formal system in terms of the standard symbolic logic meanings of the symbols. When we refer to the "meaning" of a well-formed string or theorem, we mean, the meaning under this interpretation.

Answer the following questions:

a. Using the fantasy rule derive the theorem $< P \Rightarrow P >$. Then derive the theorem $< \neg P \vee P >$, and note that you don't need an additional fantasy.

b. On page 196 of *GEB*, Hofstadter derives the theorem $<< P \wedge \neg P > \Rightarrow Q >$. What is the logical "meaning" of this theorem? Here's another way derive this: use a metafantasy to derive the theorem $< \neg Q \Rightarrow < P \Rightarrow P >>$, then use repeated applications of rule II to finally derive $<< P \wedge \neg P > \Rightarrow Q >$.

c. The statement of the absorption rule can be written as theorem of PC as

$$<<< P \wedge < P \vee Q >> \Rightarrow P > \wedge < P \Rightarrow < P \wedge < P \vee Q >>>> .$$

This consists of two theorems joined by $\wedge$. Derive the "left-hand" theorem using a single application of rule I. Also, derive the "right-hand" theorem. (Hint: start the first fantasy with P, then immediately start a metafantasy with $\neg Q$.)

d. The following well formed strings all "express" the statements of standard rules of logic:

i. $<< P \Rightarrow \neg\neg P > \wedge < \neg\neg P \Rightarrow P >>$
ii. $<<< P \Rightarrow Q > \Rightarrow < \neg Q \Rightarrow \neg P >> \wedge << \neg Q \Rightarrow \neg P > \Rightarrow < P \Rightarrow Q >>>$
iii. $<< \neg < P \vee Q > \Rightarrow < \neg P \wedge \neg Q >> \wedge << \neg P \wedge \neg Q > \Rightarrow \neg < P \vee Q >>>$
iv. $<< P \wedge < P \Rightarrow Q >> \Rightarrow Q >$

Identify each one and derive three of your choice as theorems of PC.

e. How do you know that each theorem of PC is a well formed string? Write out the truth table for three theorems of PC you've derived. Do you notice any connection between the construction of the truth table and the derivation of the string? What do you notice about the final column of these truth tables? Do you think your observation holds for all theorems of PC? Can your observation be used as a decision procedure for theorems of PC?

f. Do you think that under the standard interpretation the PC system is consistent and/or complete? In terms of your above decision procedure involving truth tables, what does it mean that PC is consistent and/or complete?