Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Fall 2011

Problem Set # 11 (due Friday 02 December 2011)

Let R be a region in $\mathbb{R}^2$, in which we use the coordinates (s,t). A **parameterized surface** in $\mathbb{R}^3$ is a mapping $\varphi : R \to \mathbb{R}^3$. The image S of φ is a **surface** in $\mathbb{R}^3$.

For example, let $f : R \to \mathbb{R}$ be a function and Γ_f its graph in $\mathbb{R}^3$, which is a the part of the surface $z = f(x,y)$ over the region R (thought of as in the x-y-plane). Then $\varphi(s,t) = (s,t,f(s,t))$ is the "standard" parameterization of Γ_f.

For another example, the cylinder $x^2 + y^2 = a^2$ has a "cylindrical" parameterization $\varphi(s,t) = (a\cos(t), a\sin(t), s)$ where $R = \{\,(s,t) \,:\, -\infty < s < \infty,\ 0 \le t \le 2\pi\,\}$.

For yet another example, the sphere $x^2 + y^2 + z^2 = a^2$ has a "spherical" parameterization $\varphi(s,t) = (a\sin(s)\cos(t), a\sin(s)\sin(t), a\cos(s))$

At each point P of a surface S in $\mathbb{R}^3$, there are two unit normal vectors to S at P. A continuous choice $\vec{n}(P)$ of such a unit normal vector at each point P of S is called an **orientation**. Amazingly, some surfaces *do not have* an orientation (a mobius strip is an example, see CM Figure 19.15).

In analogy with the velocity vector of a parameterized curve, a parameterized surface $\varphi : R \to \mathbb{R}^3$ has two different *partial derivative vectors*. Expanding out into coordinate functions $\varphi(s,t) = (\varphi_1(s,t), \varphi_2(s,t), \varphi_3(s,t))$, then define

$$\frac{\partial\varphi}{\partial s} = \frac{\partial\varphi_1}{\partial s}\vec{\boldsymbol{\imath}} + \frac{\partial\varphi_2}{\partial s}\vec{\boldsymbol{\jmath}} + \frac{\partial\varphi_3}{\partial s}\vec{\boldsymbol{k}}, \qquad \frac{\partial\varphi}{\partial t} = \frac{\partial\varphi_1}{\partial t}\vec{\boldsymbol{\imath}} + \frac{\partial\varphi_2}{\partial t}\vec{\boldsymbol{\jmath}} + \frac{\partial\varphi_3}{\partial t}\vec{\boldsymbol{k}}.$$

Then $\frac{\partial\varphi}{\partial s}$ and $\frac{\partial\varphi}{\partial t}$ are always tangent vectors to the surface. Their cross product $\frac{\partial\varphi}{\partial s} \times \frac{\partial\varphi}{\partial t}$ is then always a normal vector to the surface (or zero in some bad cases). So a parameterization basically defines a natural orientation of a surface.

Let $\vec{F}$ be a vector field on $\mathbb{R}^3$, and S an oriented surface in $\mathbb{R}^3$ with parameterization $\varphi : R \to R^3$. Then the **flux integral** of $\vec{F}$ over S is

$$\int_S \vec{F} = \int_R (\vec{F} \circ \varphi) \cdot \left(\frac{\partial\varphi}{\partial s} \times \frac{\partial\varphi}{\partial t}\right),$$

as long as the orientation chosen on S is the same as the one induced by φ. Otherwise, the flux integral is the negative of this. You should think of a flux integral as a 2-dimensional version of a line integral.

For example, if $R = \{\,(s,t) \,:\, a \le s \le b,\ c \le t \le d\,\}$ is a box, then

$$\int_S \vec{F} = \int_{t=c}^{d} \int_{s=a}^{b} \vec{F}(\varphi(s,t)) \cdot \left(\frac{\partial\varphi}{\partial s}\Big|_{(s,t)} \times \frac{\partial\varphi}{\partial t}\Big|_{(s,t)}\right) ds dt.$$

This should remind you of the formula for a line integral given a parameterized curve.

Note that in CM, they write $\vec{r}$ for a parameterized surface (what we're calling φ) and $\int_S \vec{F}\, d\vec{A}$ for the flux integral.

Reading: CM 17.5, 19.1–3

(Problems start on next page.)

1. For the following parameterized surfaces, compute $\frac{\partial \varphi}{\partial s} \times \frac{\partial \varphi}{\partial t}$.

a) Let $f : R \to \mathbb{R}$ be a function and $\varphi : R \to \mathbb{R}^3$ be the standard parameterization of its graph Γ_f defined above.
b) The cylindrical parameterization $\varphi : R \to \mathbb{R}^3$ of the cylinder $x^2 + y^2 = a^2$ defined above.
c) The spherical parameterization $\varphi : R \to \mathbb{R}^3$ of the sphere $x^2 + y^2 + z^2 = a^2$ defined above.

Compare your answers (via the formula above for the flux integral), with the boxed formulas on CM pages 980, 982, and 983.

2. CM 17.5 Problems 16

3. CM 19.1 Exercises 6, 10, 32

4. CM 19.2 Exercises 2, 8, 12, 22
Problems 26

5. CM 19.3 Exercise 2, 3, 8
Problems 10 (the point is to find a parameterization for this)

Note: Please hand in the extra credit problems on a separate sheet of paper.

6. (Extra Credit) Let R_α be the region bounded by two cylinders of radius 1 whose axes intersect with an angle α between them. Assume $0 < \alpha \leq \pi/2$. Feel free to orient one of the cylinders in a convenient position (like along the z-axis, or perhaps along the y-axis, for example). Find the volume of R_α as a function of α. You already know from class that the volume of $R_{\pi/2}$ is 16/3. You may find that making a change of coordinates will help you! What happens to the volume of R_α as $\alpha \to 0$?

7. (Extra Credit) Compute the following triple integral in two different ways:

$$\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dz dy dx$$

a) First, compute it in the order presented (you may have to remind yourself of some properties of inverse trig substitutions).
b) Second, exchange the order of integration of the outer two most integrals. Now if you were in class on Tuesday November 22nd, you should recognize the integral and know it's volume. If you skipped that day, google the phrase "steve strogatz it slices it dices" and find out (you'll need to compute this second integral then as well)!

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