Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Fall 2011

Problem Set # 6 (due Friday 14 October 2011)

Recall: Let $a \leq b$ be real numbers, $[a, b]$ the closed interval from a to b, and $\gamma : [a, b] \to \mathbb{R}^2$ a parameterized curve. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function on $\mathbb{R}^2$, then we can consider the gradient vector field $\vec{\nabla} f$ on $\mathbb{R}^2$. We have the **fundamental theorem of calculus** for line integrals:

$$\int_\gamma \vec{\nabla} f = f(\gamma(b)) - f(\gamma(a)).$$

A vector field $\vec{F}$ on R^2 is called **path-independent** or **conservative** if the line integral along a path between two points does not depend on the particular path chosen. We proved in class that $\vec{F}$ is path-independent if and only if $\vec{F} = \nabla f$ is a gradient vector field for some function $f : \mathbb{R}^2 \to \mathbb{R}$. We can view this as a test for path-independence.

Here's another test for path-independence of a vector field, called the scalar curl test. First, some notation: a region $R \subset \mathbb{R}^2$ is called **simply connected** if for every closed curve contained in R, the entire area encircled by that curve is also contained in R. Colloquially, this means that R has "no holes." For example, $\mathbb{R}^2$ is simply connected. If $\vec{F}(x, y) = F_1(x, y)\vec{\imath} + F_2(x, y)\vec{\jmath}$ is a vector field (with continuous partial derivatives, whatever that means) then the **scalar curl** of $\vec{F}$ is the function on $\mathbb{R}^2$ given by

$$\frac{\partial F_2}{\partial x}\Big|_{(x,y)} - \frac{\partial F_1}{\partial y}\Big|_{(x,y)}.$$

Finally, the **scalar curl test** says: for a vector field $\vec{F}$ on a simply connected region, if the scalar curl of $\vec{F}$ is 0 then $\vec{F}$ is path-independent in that region. You can find this in CM 18.4, pp. 954-955.

Reading: CM 18.1-4.

1. CM 18.3 Exercises 2, 10, 12, 14
Problems 21, 22, 34 (you can just draw the paths, if you'd like), 40
Do not use the scalar curl test for these problems.

2. CM 18.4 Exercises 2, 5, 6, 8, 10

3. (Extra credit) Define a vector field by

$$\vec{F}(x, y) = \left(y\sqrt{x^2 + y^2} + x^2 \ln\left|\, y + \sqrt{x^2 + y^2}\,\right|\right)\vec{\imath} + \left(x\sqrt{x^2 + y^2} + y^2 \ln\left|\, x + \sqrt{x^2 + y^2}\,\right|\right)\vec{\jmath}$$

a) Find the region of definition of $\vec{F}$. Describe and draw a picture. Explain why it is simply connected.
b) Show that the scalar curl is 0.
c) Conclude that $\vec{F}$ is path-independent.
d) Set up the line integral of $\vec{F}$ along the circle or radius 1 centered at the origin using the standard parameterization $(\cos(t), \sin(t)$ for $0 \leq t \leq 2\pi$.
Note: you cannot appeal to the path-independence of $\vec{F}$ to conclude that this line integral is zero, since the path crosses over the region where $\vec{F}$ is not defined!
e) Use a computer to sketch the integrand of this line integral.
f) Using the computer sketch, argue that the line integral (around the unit circle) is indeed zero!
g) Use the symmetry of the vector field around the line $y = x$ to verify your conclusions from looking at the computer sketch.

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