Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Fall 2011

Problem Set # 7 (due Friday 21 October 2011)

Material: Let $a \leq b$ be real numbers, and let $l(x)$ and $u(x)$ be continuous real functions satisfying $l(x) \leq u(x)$ for all $x \in [a,b]$. Consider $[a,b] \subset \mathbb{R}^2$ as an interval on the x-axis and the region R of $\mathbb{R}^2$ "above the interval $[a,b]$ and between the graphs of $l(x)$ and $u(x)$" defined by

$$R = \{\, (x,y) \in \mathbb{R}^2 \,:\, a \leq x \leq b,\, l(x) \leq y \leq u(x) \,\}.$$

For example, if $l(x) = c$ and $u(x) = d$ are constant functions, then R is the box with corners (a,c), (b,c), (a,d), and (b,d). If $a = 0$, $b = 1$, $l(x) = 0$ and $u(x) = x$, then R is a triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$. If $a = -1$, $b = 1$, $l(x) = -\sqrt{1-x^2}$, and $u(x) = \sqrt{1-x^2}$, then R is a circle of radius 1 centered at the origin.

For each region R in $\mathbb{R}^2$ and each continuous function $f : R \to \mathbb{R}$, we introduced a multivariable integral of f over R

$$\int_R f$$

satisfying the property that if $f(x,y) \geq 0$, then $\int_R f$ is the volume of the region above R (thought of in the x-y-plane) and below the graph of f.

If the region R in $\mathbb{R}^2$ is defined by a, b, $l(x)$, and $u(x)$ as above, then we write the multivariable integral of f over R as

$$\int_R f = \int_{x=a}^{b} \int_{y=l(x)}^{u(x)} f(x,y)\, dy\, dx$$

and we can perform "iterated integration" just as in Fubini's theorem: first find an antiderivative of $f(x,y)$ with respect to y, i.e. a function $g(x,y)$ so that $\frac{\partial g}{\partial y} = f(x,y)$, then

$$\begin{aligned} \int_{x=a}^{b} \int_{y=l(x)}^{u(x)} f(x,y)\, dy\, dx &= \int_{x=a}^{b} \left(g(x,y) \Big|_{y=l(x)}^{u(x)} \right) dx \\ &= \int_{x=a}^{b} \big(g(x,u(x)) - g(x,l(x)) \big)\, dx \end{aligned}$$

and then integrate with respect to x in the usual way.

We can also define regions R in $\mathbb{R}^2$ "in the other direction" by

$$R = \{\, (x,y) \in \mathbb{R}^2 \,:\, c \leq y \leq d,\, p(y) \leq x \leq q(y) \,\},$$

where $c \leq d$ and $p(y) \leq q(y)$ for all $y \in [c,d]$. Then we write the multivariable integral of f over R as

$$\int_R f = \int_{y=c}^{d} \int_{x=p(y)}^{q(y)} f(x,y)\, dx\, dy$$

and we can perform "iterated integration" in the same way, by first finding an antiderivative of $f(x,y)$ with respect to x. You can find worked out examples of this in CM 16.2. This homework assignment is a mild introduction to this concept.

Reading: CM 16.1-2.

0. (Don't hand these in, but make sure you know how to do them.) CM 16.1 Problems 9–24

1. CM 16.1 Problem 26

2. CM 16.2 Exercises 2, 4, 6, 8, 10, 12, 14, 16, 18
For 12–16 write the iterated integral in both forms $\int_{x=?}^{?} \int_{y=?}^{?} f(x,y)\, dydx$ and $\int_{y=?}^{?} \int_{x=?}^{?} f(x,y)\, dxdy$.
For 16 you might have to write a sum of two integrals for one of these forms.

3. CM 16.2 Problems 29

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