Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Spring 2010

Problem Set # 6 (due Wednesday 17 March 2010)

Recall: Let $a \leq b$ be real numbers, $[a, b] \subset \mathbb{R}$ the closed interval from a to b, $\gamma : [a, b] \to \mathbb{R}^2$ a parameterized curve, and $\vec{F}$ a vector field in $\mathbb{R}^2$, then the line integral of $\vec{F}$ along γ is computed by the definite integral

$$\int_\gamma \vec{F} = \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt.$$

If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function on $\mathbb{R}^2$, then we can consider the gradient vector field ∇f on $\mathbb{R}^2$. We have the **fundamental theorem of calculus** for line integrals:

$$\int_\gamma \nabla f = f(\gamma(b)) - f(\gamma(a)).$$

A vector field $\vec{F}$ is called **path-independent** or **conservative** if the line integral along a path between two points does not depend on the particular path chosen. We proved in class that $\vec{F}$ is path-independent if and only if $\vec{F} = \nabla f$ is a gradient vector field for some function $f : \mathbb{R}^2 \to \mathbb{R}$. We can view this as a test for path-independence.

Here's another test for path-independence of a vector field, called the **curl test**. First, some notation: a region $R \subset \mathbb{R}^2$ is called **simply connected** if for every closed curve contained in R, the entire area encircled by that curve is also contained in R. Colloquially, this means that R has "no holes." If $\vec{F}(x, y) = F_1(x, y)\vec{\imath} + F_2(x, y)\vec{\jmath}$ is a vector field (with continuous partial derivatives, whatever that means) then the **scalar curl** of $\vec{F}$ is the function $\frac{\partial F_2}{\partial x}\Big|_{(x,y)} - \frac{\partial F_1}{\partial y}\Big|_{(x,y)}$. Finally, the curl test says: for a vector field $\vec{F}$ on a simply connected region, if the scalar curl of $\vec{F}$ is 0 then $\vec{F}$ is path-independent. You can find this in CM 18.4, pp. 954-955.

Reading: CM 18.1-4.

1. CM 18.3 Exercises 2, 10, 12, 14, 16
Problems 21, 22, 34, 36, 40
You cannot use the curl test for these problems.

2. CM 18.4 Exercises 2, 6, 8, 13

3. * Define a vector field by

$$\vec{F}(x, y) = \frac{-y}{x^2 + y^2}\vec{\imath} + \frac{x}{x^2 + y^2}\vec{\jmath}$$

a) Parameterize a circle or radius 1 centered at the origin and calculate the (circulation) line integral of $\vec{F}$ along it.

b) For each real number a, parameterize a circle or radius 1 centered at $(a, 0)$ and calculate the (circulation) line integral of $\vec{F}$ along it (write your answer in terms of a). Just set up the integral.

c) What happens when $a = 1$? Can you evaluate the line integral? Does your line integral have a meaning? Does your formula have a meaning? Explain.

d) If you evaluate your integral (from part *c)*) for a general a, show your work and you will get extra extra credit. Otherwise, use a computer to calculate the antiderivative of the integrand, and evaluate the integral by hand. Simplify as much as you can.

e) How does the integral depend on a? Explain in words what is happening as a increases.

f) Is $\vec{F}$ path-independant on $\mathbb{R}^2$? Prove that $\vec{F}$ is path-independant on the open right half-plane $\{\, (x, y) \in \mathbb{R} : x > 0 \,\}$ by finding a potential function for $\vec{F}$ on that region. Also find a potential function for $\vec{F}$ on the open upper half-plane $\{\, (x, y) \in \mathbb{R} : y > 0 \,\}$.