Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Spring 2012

Final (Tue 08 May 2012, 08:30–11:00 am, MSC 301) Study Guide and Practice Exam

Section I

1. Determine whether the following statements are always true (T), sometimes true/sometimes false (T/F), or always false (F). You do not need to justify your answer.

a) T
b) T
c) F
d) T/F
e) T/F
f) F
g) T

2. Let $P = (1,2,3)$, $Q = (3,5,7)$, and $R = (2,5,3)$.

- A normal vector at P through the plane through P, Q, and R is $\overrightarrow{PQ} \times \overrightarrow{PR} = -12\vec{\imath} + 4\vec{\jmath} + 3\vec{k}$. So an equation for the plane is $-12x + 4y + 3z = 5$.
- A unit normal vector to this plane: $\frac{12}{13}\vec{\imath} - \frac{4}{13}\vec{\jmath} - \frac{3}{13}\vec{k}$.
- The angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$: $\cos^{-1}\left(\frac{11}{\sqrt{290}}\right)$.
- The area of the triangle with vertices P, Q, and R: $\frac{13}{2}$.
- The shortest distance from R to the line through P and Q: $\frac{13}{\sqrt{29}}$, hint use the fact that you already know the area of the triangle, this distance is the "height" if the base is along the line from P to Q.

3. Finding tangent planes.

a) Let $S = \{x^2 - y^2 = 3\} \subset \mathbb{R}^2$ and $P = (2,1)$.
- A normal vector to the curve S at the point P: $4\vec{\imath} - 2\vec{\jmath}$.
- A parameterization of the tangent line to S at P: (A fact about vectors in the plane: the vector $-b\vec{\imath} + a\vec{\jmath}$ is perpendicular to the vector $a\vec{\imath} + b\vec{\jmath}$.) So $2\vec{\imath} + 4\vec{\jmath}$ is a vector perpendicular to $4\vec{\imath} - 2\vec{\jmath}$, so $2\vec{\imath} + 4\vec{\jmath}$ is tangent to the curve at $(2,1)$, then we can parameterize the tangent line as $\gamma(t) = (2+2t, 1+4t)$.

b) Let $S = \{z^2 - 2xyz = x^2 + y^2\}$ and $P = (1,2,-1)$.
- A normal vector to the surface S at the point P: $2\vec{\imath} - 2\vec{\jmath} - 6\vec{k}$.
- An equation of the tangent plane to S at P: $2x - 2y - 6z = 4$.

4. Let $f(x,y) = \frac{x-y}{x^2+1}$ and $P = (1,1,0)$.

- A normal vector to the graph of f at P: $\frac{1}{2}\vec{\imath} - \frac{1}{2}\vec{\jmath} - \vec{k}$.
- An equation of the tangent plane of the graph of f at P: $x - y - 2z = 0$.
- A tangent vector to the graph of f at P pointing in the direction of steepest ascent: $\frac{1}{2}\vec{\imath} - \frac{1}{2}\vec{\jmath} + \frac{1}{2}\vec{k}$.

Section II

5. Determine whether the following statements are always true, sometimes true/sometimes false, or always false. You do not need to justify your answer.

a) F
b) F

c) T/F
d) F
e) T/F

6. Compute the following line integrals $\int_\gamma \vec{F}$.

a) $\frac{98}{3}$
b) $\frac{14}{3}$
c) 0
d) $\frac{1}{2}$

7. For each vector field $\vec{F}$ find (and sketch) the region of definition, determine if the region is simply connected, and determine if $\vec{F}$ is path-independent.

a) Region: $y \neq 0$, i.e. off the x-axis, is simply connected. Path-independent by scalar curl test.
b) Region: $x^2 + y^2 > 1$, i.e. off the disk of radius 1 centered at the origin, not simply connected. Path-independent, this fails all tests, you need to search for the potential function directly!
c) $x^2 > 1$ and $y^2 > 1$, i.e. $|x| > 1$ and $|y| > 1$, i.e. off the square of side length 2 centered at the origin, not simply connected. Not path-independent by scalar curl test.
d) $y \leq 1 - x^2$, i.e. underneath a downward facing parabola, simply connected. Not path-independent by scalar curl test.

8. Change the order of integration of $\int_{z=0}^{1} \int_{y=0}^{z^2} \int_{x=y}^{2} f(x, y, z)\, dxdydz$ in all the other 5 possible ways. You have to draw 5 pictures! You change two adjacent integrals at a time, pretending that the other one is invisible, and working the the appropriate plane.

- Changing the y and z integrals means drawing a picture in the y-z-plane of the limits on the y and z integrals, pretending that the x integral isn't there. Doing this we get

$$\int_{y=0}^{1} \int_{z=\sqrt{y}}^{1} \int_{x=y}^{2} f(x, y, z)\, dxdzdy.$$

- From here, changing x and z, we see that the limits of both the x and z integrals only depend on constants and y, which because it is a constant at this point (but to be integrated at the end, to be sure), the limits are just constant with respect to x and y, so we're integrating over a "box." By Fubini's theorem,

$$\int_{y=0}^{1} \int_{x=y}^{2} \int_{z=\sqrt{y}}^{1} f(x, y, z)\, dzdxdy.$$

- From here, changing x and y, we see that we have to break the region into two pieces

$$\int_{x=0}^{1} \int_{y=0}^{x} \int_{z=\sqrt{y}}^{1} f(x, y, z)\, dzdydx + \int_{x=1}^{2} \int_{y=0}^{1} \int_{z=\sqrt{y}}^{1} f(x, y, z)\, dzdydx.$$

- Now going back to the original integral and changing the x and y integrals, we see that for fixed z between 0 and 1, the picture of $0 \leq y \leq z^2$, $y \leq x \leq 2$ has two pieces when x is independent. So changing x and y breaks the integral into two pieces

$$\int_{z=0}^{1} \int_{x=0}^{z^2} \int_{y=0}^{x} f(x, y, z)\, dydxdz + \int_{z=0}^{1} \int_{x=z^2}^{2} \int_{y=0}^{z^2} f(x, y, z)\, dydxdz.$$

- Finally, only the "$dydzdx$" integral remains, which we get from changing the x and z integrals from above. For each integral, remembering that z is a constant with respect to the x and y integrals, draw a picture: the first is the triangle and the second is a square. Changing finally gives:

$$\int_{z=0}^{1} \int_{y=0}^{z^2} \int_{x=y}^{z^2} f(x, y, z)\, dxdydz + \int_{z=0}^{1} \int_{y=0}^{z^2} \int_{x=z^2}^{2} f(x, y, z)\, dxdydz.$$

9. Computing volumes of general solids.

a) Let a, b, c be positive numbers. Find the volume of the region between the coordinate planes and the plane $ax + by + cz = 1$.

Here's one method: find above what part of the x-y-plane does the given plane lay then integrate $z = \frac{1}{c}(1 - ax - by)$. Note that if any of a, b, or c is zero, then this plane is parallel to one axis, so the volume in question is infinite. This region in the x-y-plane is bounded by the x-axis, the y-axis, and where the plane intersects the x-y-plane, i.e. by setting $z = 0$, this gives $ax + by = 1$. This line in the x-y-plane intersects the x-axis at $x = \frac{1}{a}$, so the region on the x-axis can be given by $0 \leq x \leq \frac{1}{a}$, $0 \leq y \leq \frac{1}{b}(1 - ax)$. Finally the integral is

$$\int_{x=0}^{\frac{1}{a}} \int_{y=0}^{\frac{1}{b}(1-ax)} \frac{1}{c}(1 - ax - by)\, dydx = \frac{1}{6abc}$$

b) Find the volume of the solid formed by drilling out a cylindrical hole of radius a through the center of a sphere of radius R. Of course, we assume that $0 \leq a \leq R$.

We'll use cylindrical coordinates and put the sphere with center at the origin and hole being drilled out along the z-axis. The radius r can go from the radius of the cylinder a to the radius of the sphere R while the height z goes from the bottom of the sphere to the top. This gets set up as:

$$\int_{\theta=0}^{2\pi} \int_{r=a}^{R} \int_{z=-\sqrt{R^2-r^2}}^{-\sqrt{R^2-r^2}} r\, drdzd\theta = \frac{4\pi}{3}(R^2 - a^2)^{3/2}.$$

Or in the other direction, the radius R forms a right triangle with the height of the shape (from the x-y-axis) as one side and the hole radius a as one side, so this height is $\sqrt{R^2 - a^2}$. Then the integral that needs to be set up is:

$$\int_{\theta=0}^{2\pi} \int_{z=-\sqrt{R^2-a^2}}^{\sqrt{R^2-a^2}} \int_{r=a}^{\sqrt{R^2-z^2}} r\, drdzd\theta = \frac{4\pi}{3}(R^2 - a^2)^{3/2}.$$

Do which ever you like better!

Section III

10. Determine whether the following statements are always true, sometimes true/sometimes false, or always false. You do not need to justify your answer.

a) T/F. If f is purely a function of x and g is purely a function of y, then this statement is true. If both f and g depend on both x and y, then this statement need not be true.

b) T. Both integrals represent the integral of f over the region below the plane $x + y + z = 1$ and above the x-y-axis.

c) T/F. This is true for the function $f(x, y) = 1/2$, but in general, using the change of variables theorem with $x = u$, $y = 2v$, we see that the box $0 \leq u \leq 2$, $0 \leq v \leq 1$ in the u-v-plane is mapped to the box $0 \leq x \leq 2$, $0 \leq y \leq 2$ in the x-y-plane. You should check that the change of variables theorem does not take one integrand to the other under this mapping! So probably it isn't true in general. Check with a simple example: $\int_0^1 \int_0^2 y\, dxdy = 1$ yet $\int_0^2 \int_0^2 2y\, dxdy = 4 \neq 2$.

d) T. For example, if R is a region in the x-y-plane, and $f(x, y)$ is a nonnegative function, then $\int_R f$ is the volume of the region below the graph of f and above R.

e) T. The flux of each face will cancel with the flux of the opposite face.

f) T. You can use the divergence theorem.

11. Let R_a be the upper hemisphere of the solid sphere of some radius $a > 0$ centered at the origin. In spherical coordinates, we have

$$\begin{aligned}\int_{R_a} z\, dxdydz &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{a} \rho\cos(\phi)\,\rho^2\sin(\phi)\,d\rho d\phi d\theta \\ &= 2\pi\int_{\phi=0}^{\pi/4}\frac{a^4}{4}\cos(\phi)\sin(\phi)\,d\phi = \frac{\pi a^4}{2}\,\frac{1}{2}\sin^2(\phi)\Big|_{\phi=0}^{\pi/4} = \frac{\pi a^4}{4}.\end{aligned}$$

Side note for physics people: this says that the z-coordinate of the center of mass of R_a (thought of with constant density) is $\frac{3}{2\pi a^3}\frac{\pi a^4}{4}$. So the center of mass is at the point $(0, 0, \frac{3a}{8})$. Makes sense!

12. Let R be the polyhedral region with vertices $(0, 0)$, $(2, 5)$, $(1, 2)$, and $(3, 7)$. To calculate $\int_R xy^2\, dxdy$, we use the change of coordinates $\Phi(s, t) = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} = (s + 2t, 2s + 5t) = (x(s, t), y(s, t))$. So this makes $x(s, t) = s + 2t$ and $y(s, t) = 2s + 5t$. This transforms the unit square into R. So now you're integrating the function $(s + 2t)(2s + 5t)^2$ over the unit square in the s-t-plane. You can calculate this integral, which turns out to be 29.

13. Let S be the region contained in the closed curve $x^2 - xy + y^2 = 1$. Compute $\int_S xy\, dxdy$ by using the change of coordinates $x = s - \frac{1}{\sqrt{3}}t$, $y = s + \frac{1}{\sqrt{3}}t$. Notice that substituting in the change of variables yields:

$$1 = \left(s - \frac{1}{\sqrt{3}}t\right)^2 - \left(s - \frac{1}{\sqrt{3}}t\right)\left(s + \frac{1}{\sqrt{3}}t\right) + \left(s + \frac{1}{\sqrt{3}}t\right)^2 = s^2 + t^2$$

so that the unit disk R in the s-t-plane maps under $\Phi(s, t) = (s - \frac{1}{\sqrt{3}}t, s + \frac{1}{\sqrt{3}}t)$ to the region S in the x-y-plane. Now to compute the jacobian matrix and its determinant

$$J_{(s,t)}\Phi = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{\sqrt{3}} \\ 1 & \frac{1}{\sqrt{3}} \end{pmatrix}, \qquad \det J_{(s,t)}\Phi = \frac{2}{\sqrt{3}}.$$

Finally, the integral is, using the change of variables theorem:

$$\int_{\Phi(R)} xy\, dxdy = \int_R (s - \frac{1}{\sqrt{3}}t)(s + \frac{1}{\sqrt{3}}t)\,\frac{2}{\sqrt{3}}\,dsdt = \frac{2}{\sqrt{3}}\int_R (s^2 - \frac{1}{3}t^2)\,dsdt = \frac{\pi}{3\sqrt{3}},$$

by integrating over the unit circle (at this point you probably want to switch to polar coordinates)!

14. Let S be the surface given by the cyclinder $x^2 + y^2 = 1$ above the x-y-plane of height 4.

a) A parameterization for S, including limits: $\varphi(t, s) = (\cos(t), \sin(t), s)$ where $0 \le t \le 2\pi$ and $0 \le s \le 4$. (See CM chapter 17.5 for more information on parameterizing surfaces!)

b) First, we can calculate $\frac{\partial\varphi}{\partial s}$ and $\frac{\partial\varphi}{\partial t}$:

$$\begin{aligned}\frac{\partial\varphi}{\partial s} &= \frac{\partial\varphi_1}{\partial s}\vec{\imath} + \frac{\partial\varphi_2}{\partial s}\vec{\jmath} + \frac{\partial\varphi_3}{\partial s}\vec{k} = \vec{k} \\ \frac{\partial\varphi}{\partial t} &= \frac{\partial\varphi_1}{\partial t}\vec{\imath} + \frac{\partial\varphi_2}{\partial t}\vec{\jmath} + \frac{\partial\varphi_3}{\partial t}\vec{k} = -\sin(t)\vec{\imath} + \cos(t)\vec{\jmath}.\end{aligned}$$

Then we have

$$\frac{\partial\varphi}{\partial s} \times \frac{\partial\varphi}{\partial t} = \cos(t)\vec{\imath} + \sin(t)\vec{\jmath}.$$

Then the flux integral of $\vec{F}(x,y,z) = x\vec{\imath} + y\vec{\jmath}$ along S is

$$\begin{aligned}
\int_S \vec{F} &= \int_{t=0}^{2\pi} \int_{s=0}^{4} \vec{F}(\cos(t), \sin(t), s) \cdot \left(\frac{\partial \varphi}{\partial s} \times \frac{\partial \varphi}{\partial t} \right) ds dt \\
&= \int_{t=0}^{2\pi} \int_{s=0}^{4} (\cos(t)\vec{\imath} + \sin(t)\vec{\jmath}) \cdot (\cos(t)\vec{\imath} + \sin(t)\vec{\jmath})\, ds dt \\
&= \int_{t=0}^{2\pi} \int_{s=0}^{4} ds dt = 8\pi.
\end{aligned}$$

Emory University, Department of Mathematics & CS, 400 Dowman Dr NE W401, Atlanta, GA 30322
E-mail address: auel@mathcs.emory.edu