EMORY UNIVERSITY DEPARTMENT OF MATHEMATICS & CS
Math 211 Multivariable Calculus
Spring 2012

Problem Set # 7-8 (due Friday 30 March 2012)

Material: Let $a \le b$ be real numbers, and let $l(x)$ and $u(x)$ be continuous real functions satisfying $l(x) \le u(x)$ for all $x \in [a,b]$. Consider $[a,b] \subset \mathbb{R}^2$ as an interval on the x-axis and the region R of $\mathbb{R}^2$ "above (and possible below) the interval $[a,b]$ and between the graphs of $l(x)$ and $u(x)$" defined by

$$R = \{\, (x,y) \in \mathbb{R}^2 \,:\, a \le x \le b,\, l(x) \le y \le u(x) \,\}.$$

For example, if $l(x) = c$ and $u(x) = d$ are constant functions, then R is the box with corners (a,c), (b,c), (a,d), and (b,d). If $a = 0$, $b = 1$, $l(x) = 0$ and $u(x) = x$, then R is a triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$. If $a = -1$, $b = 1$, $l(x) = -\sqrt{1-x^2}$, and $u(x) = \sqrt{1-x^2}$, then R is a circle of radius 1 centered at the origin.

For each region R in $\mathbb{R}^2$ and each continuous function $f : R \to \mathbb{R}$, we introduced a multivariable integral of f over R

$$\int_R f$$

satisfying the property that if $f(x,y) \ge 0$, then $\int_R f$ is the volume of the region above R (thought of in the x-y-plane) and below the graph of f.

If the region R in $\mathbb{R}^2$ is defined by a, b, $l(x)$, and $u(x)$ as above, then we write the multivariable integral of f over R as

$$\int_R f = \int_{x=a}^{b} \int_{y=l(x)}^{u(x)} f(x,y)\, dy\, dx$$

and we can perform "iterated integration" just as in Fubini's theorem: first find an antiderivative of $f(x,y)$ with respect to y, i.e. a function $g(x,y)$ so that $\frac{\partial g}{\partial y} = f(x,y)$, then

$$\begin{aligned} \int_{x=a}^{b} \int_{y=l(x)}^{u(x)} f(x,y)\, dy\, dx &= \int_{x=a}^{b} \left(g(x,y)\Big|_{y=l(x)}^{u(x)} \right) dx \\ &= \int_{x=a}^{b} \big(g(x,u(x)) - g(x,l(x)) \big)\, dx \end{aligned}$$

and then integrate with respect to x in the usual way.

We can also define regions R in $\mathbb{R}^2$ "in the other direction" by

$$R = \{\, (x,y) \in \mathbb{R}^2 \,:\, c \le y \le d,\, p(y) \le x \le q(y) \,\},$$

where $c \le d$ and $p(y) \le q(y)$ for all $y \in [c,d]$. Then we write the multivariable integral of f over R as

$$\int_R f = \int_{y=c}^{d} \int_{x=p(y)}^{q(y)} f(x,y)\, dx\, dy$$

and we can perform "iterated integration" in the same way, by first finding an antiderivative of $f(x,y)$ with respect to x. You can find worked out examples of this in CM 16.2. This homework assignment is a mild introduction to this concept.

Reading: CM 16.1-2.

0. (Don't hand these in, but make sure you understand the point.) CM 16.1 Problems 9–24

1. CM 16.1 Problem 26 (Explain a bit in English.)

2. CM 16.2 Exercises 3, 4, 6, 8, 10, 12, 14, 16, 18
For 12–16 write the iterated integral in both forms $\int_{x=?}^{?} \int_{y=?}^{?} f(x,y)\, dydx$ and $\int_{y=?}^{?} \int_{x=?}^{?} f(x,y)\, dxdy$.
For 16 you might have to write a sum of two integrals for one of these forms.

3. CM 16.2 Problems 32, 34, 37, 38, 42, 44, 48

EMORY UNIVERSITY, DEPARTMENT OF MATHEMATICS & CS, 400 DOWMAN DR NE W401, ATLANTA, GA 30322
E-mail address: auel@mathcs.emory.edu