Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Spring 2012

Problem Set # 9 (due Friday 13 April 2012)

Material: Let $a \le b$ be real numbers, $l_y(x)$ and $u_y(x)$ be continuous single-variable real valued functions satisfying $l_y(x) \le u_y(x)$ for all x in $[a, b]$, and $l_z(x, y)$ and $u_z(x, y)$ be continuous two-variable real valued functions satisfying $l_z(x, y) \le l_z(x, y)$ for all points in the region $\{ (x, y) : a \le x \le b,\ l_y(x) \le y \le u_u(x) \}$ of the x-y-plane. Finally let

$$R = \{ (x, y, z) \in \mathbb{R}^3 : a \le x \le b,\ l_y(x) \le y \le u_y(x),\ l_z(x, y) \le z \le u_z(x, y) \}$$

and $f : R \to \mathbb{R}$ be a continous function on the region R. Then there's a multivariable integral of f over R, which is equal to the iterated integral

$$\int_R f = \int_{x=a}^{b} \int_{y=l_y(x)}^{u_y(x)} \int_{z=l_z(x,y)}^{u_z(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

satisfying the following property: thinking of f as the "density" function of the solid region R, then $\int_R f$ is the mass of R, in particular, $\int_R 1$ is the volume of R.

Reading: CM 16.3-5.

1. CM 16.3 Exercises 2 (note: a, b, c are arbitrary real constants; your answer should be expressed in terms of them!), 6, 8, 10

Problem 16 (note that above the first quadrant ($x \ge 0$ and $y \ge 0$) of the x-y-plane, the plane $3x + 4y + z = 6$ is below the plane $2x + 2y + z = 6$ and you are interested in finding the volume of the region *between* the two planes over the given triangle $x + y \le 1$ in the x-y-plane),

Problem 18 ("Under the sphere" really means "Inside the sphere").

2. CM 16.4 Exercises 6, 8, 12, 14, 16

Problems 20, 22

3. (Extra credit) During the course of this problem, you will compute the "improper integral"

$$G = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

which is defined as the limit $\lim_{r\to\infty} G(r)$, where $G(r) = \int_{-r}^{r} e^{-x^2}\, dx$ for $r \ge 0$. Let $f(x, y) = e^{-x^2-y^2}$.

a) Let $R(r)$ be the box with corners $(r, -r)$, (r, r), $(-r, r)$, and $(-r, -r)$. Use Fubini's Theorem to show that $G(r)^2 = \int_{R(r)} f$.

b) Let $C(r)$ be the disk of radius r centered at the origin. Show that

$$\int_{C(r)} f \le \int_{R(r)} f \le \int_{C(\sqrt{2}r)} f.$$

c) Change to polar coordinates to compute $\int_{C(r)} f$ and $\int_{C(\sqrt{2}r)} f$.

d) Now calculate the limits

$$\lim_{r\to\infty} \int_{C(r)} f, \qquad \text{and} \qquad \lim_{r\to\infty} \int_{C(\sqrt{2}r)} f.$$

e) From this, what do you conclude is the value of $\lim_{r\to\infty} G(r)^2$.

f) Finally, what do you conclude is the value of G?

Note: to make the final two steps really rigorous, you'll have to take Math 411 (or they might cover it in Math 250, depending on the instructor)!