Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Spring 2012

Midterm # 2 (Tue 03 Apr 2012) Practice Exam

Directions: You will have 1 hour and 10 minutes for the midterm exam. No electronic devices will be allowed. No notes will be allowed. Should you need them, you will have the following formulas and definitions during the exam:

Let $\gamma : [a, b] \to \mathbb{R}^2$ be a parameterized curve and $\vec{F}$ a vector field on $\mathbb{R}^2$. Then

$$\int_\gamma \vec{F} = \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt.$$

If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function on $\mathbb{R}^2$, the **fundamental theorem of calculus** for line integrals says

$$\int_\gamma \nabla f = f(Q) - f(P)$$

if γ starts at $P = \gamma(a)$ and ends at $Q = \gamma(b)$.

A vector field $\vec{F}$ defined on a region $R \subset \mathbb{R}^2$ is called **path-independent** if given any two points in R, the line integral along a path between the two points does not depend on the particular path chosen.

The **scalar curl** of a vector field $\vec{F}(x, y) = F_1(x, y)\vec{\imath} + F_2(x, y)\vec{\jmath}$ is the function $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$.

Practice problems: The following assortment of problems is inspired by what will appear on the midterm exam, but is not necessarily representative of the length of the midterm exam. The actual midterm exam will contain many fewer parts to each problem.

1. Clearly state all the four tests we have for path-independence. There are two tests for path-independence:

- having a potential function, i.e. being a gradient field,
- having a simply connected region of definition and zero scalar curl

and two tests for non-path-independence:

- non-zero integral around a closed curve,
- non-zero scalar curl.

Keep in mind the examples of vector fields $\vec{F}$ from class:

- having a non simply connected region of definition, zero scalar curl, and are not path-independent;
- having a non simply connected region of definition, zero scalar curl, and are path-independent;
- having some line integrals around closed curves zero and some nonzero.

You might try coming up with your own examples!

2. For each of the following vector fields $\vec{F}$, determine: its region of definition, whether or not its region of definition is simply connected, and whether it is path-independent on its region of definition. If it is path-independent on its region of definition, write a potential function; if not, appeal to one of the tests to prove that no potential function exists on its region of definition.

a) $\vec{F} = x \sin(x)\vec{\imath} + y \ln(y)\vec{\jmath}$

b) $\vec{F} = x(\ln(x^2) + 1) \sin(y)\vec{\imath} + x^2 \ln(x) \cos(y)\vec{\jmath}$

c) $\vec{F} = 2xy^2\vec{\imath} + x^2\vec{\jmath}$

d) $\vec{F} = (y^2 + y + 2xy)\vec{\imath} + (x^2 + x + 2xy)\vec{\jmath}$

e) $\vec{F} = \frac{1+y}{y^2+2y+2-2x+x^2}\vec{\imath} + \frac{1-x}{y^2+2y+2-2x+x^2}\vec{\jmath}$

f) $\vec{F} = \frac{x}{x^2+y^2}\vec{\imath} + \frac{y}{x^2+y^2}\vec{\jmath}$

3. For the following vector fields $\vec{F}$ and curves γ, calculate the line integral $\int_\gamma \vec{F}$.

a) $\vec{F} = x^2\vec{\imath} + y^2\vec{\jmath}$ and γ is the line segment starting at the origin and ending at the point $(2,3)$.

b) $\vec{F} = (x^2+y)\vec{\imath} + y^2\vec{\jmath}$ and γ is the line segment starting at the origin and ending at the point $(2,3)$.

c) $\vec{F} = (y + e^{\sin(x)})\vec{\imath} + (x + \sqrt{y^2+1})\vec{\jmath}$ and γ is the unit circle going counter clockwise

d) $\vec{F} = \frac{1}{1+y^2+2xy+x^2}\vec{\imath} + \frac{1}{1+y^2+2xy+x^2}\vec{\jmath}$ and γ is the unit circle going counter clockwise.

e) $\vec{F} = (x^2+y^2)\vec{\imath} + xy\vec{\jmath}$ and γ is the unit circle going counter clockwise.

4. Let $\vec{F}(x,y) = \vec{\imath} + \cos(x)\vec{\jmath}$ be a vector field on $\mathbb{R}^2$.

a) Is $\vec{F}$ path-independent? State which test you are using.

b) Let $\gamma(t) = (t,0)$ and $\delta(t) = (t, \sin(t))$, for $0 \le t \le \pi$. Compute the line integrals $\int_\gamma \vec{F}$ and $\int_\delta \vec{F}$. Does your calculation reaffirm your answer to the previous part?

5. Match the integral with the appropriate region of integration:

a) The triangle with vertices $(0,0), (2,0), (0,1)$.
b) The triangle with vertices $(0,0), (0,2), (1,0)$.
c) The triangle with vertices $(0,0), (2,0), (2,1)$.
d) The triangle with vertices $(0,0), (1,0), (1,2)$.

a) $\int_0^1 \int_0^{2-2x} f(x,y)\,dydx$
b) $\int_0^1 \int_0^{2-2y} f(x,y)\,dxdy$
c) $\int_0^1 \int_0^{2x} f(x,y)\,dydx$
d) $\int_0^1 \int_{2y}^{2} f(x,y)\,dxdy$

Switch the order of integral in each integral.

6. Calculate the double integral

$$\int_{x=-1}^{1} \int_{y=0}^{1-x^2} \sin\big(\pi(1-y)^{3/2}\big)\,dydx.$$

Hint: you may need to switch the order of integration.

7. Calculate the volume of the region under the plane $3x+4y+6z = 12$ and in the first octant, i.e. where $x \ge 0$, $y \ge 0$, and $z \ge 0$.

8. Calculate the volume of the region under the graph of $f(x,y) = 1 - 8x - y^2$ and bounded by the x-y-plane and the x-y-plane (i.e. where $x \ge 0$ and $z \ge 0$).

9. Calculate the double integral $\int_R f$, where $f(x,y) = 8xy$ and R is the region (in the plane) between circles of radius 1 and 2 in the first quadrant. Hint: you may want to use polar coordinates.

10. Calculate the double integral $\int_R f$, where $f(x,y) = e^{x^2+y^2}$ and R is the unit disk centered at the origin.

Emory University, Department of Mathematics & CS, 400 Dowman Dr NE W401, Atlanta, GA 30322
E-mail address: auel@mathcs.emory.edu