Emory University Department of Mathematics & CS
**Math 211 Multivariable Calculus**
Spring 2012

Midterm # 2 (Thu 05 Apr 2012) Practice Exam Solution Guide

**Practice problems:** The following assortment of problems is inspired by what will appear on the midterm exam, but is not necessarily representative of the length of the midterm exam. The actual midterm exam will contain many fewer parts to each problem.

**1.** Keep in mind the examples of vector fields $\vec{F}$ from class:

**Solution.**

- $\vec{F}(x,y) = \dfrac{-y}{x^2+y^2}\vec{\imath} + \dfrac{x}{x^2+y^2}\vec{\jmath}$ has a non simply connected region of definition (namely $\mathbb{R}^2$ with the origin removed), zero scalar curl, and is not path-independent (since we checked that the line integral around the unit circle is nonzero), yet the line integral around any closed loop *not* enclosing the origin is zero;
- $\vec{F}(x,y) = \dfrac{x}{x^2+y^2}\vec{\imath} + \dfrac{y}{x^2+y^2}$ has a non simply connected region of definition (again $\mathbb{R}^2$ with the origin removed), zero scalar curl, and is path-independent (indeed, $f(x,y) = \frac{1}{2}\ln(x^2+y^2)$ is a potential function);

**2.** For each of the following vector fields, determine where it is defined and whether it is a gradient field on its region of definition. If it is a gradient field on its region of definition, write a potential function; if not, appeal to one of the tests to prove that no potential function exists on its region of definition.

*a)* $\vec{F} = x\sin(x)\vec{\imath} + y\ln(y)\vec{\jmath}$

**Solution.** The only issue may come from $\ln(y)$, which is only defined for $y > 0$. So the region of definition is $R = \{\,(x,y) \,:\, y > 0\,\}$, which is the upper half plane (so is simply connected). We notice right away that $\vec{F}$ is *separable* (remember, this means that $F_1$ is only a function of $x$ and $F_2$ is only a function of $y$), so it is automatically a gradient field regardless of its region of definition. The potential function is the sum of the antiderivatives of $F_1$ and $F_2$, which is $f(x,y) = -x\cos(x) + \sin(x) + \frac{1}{2}y^2\ln(y) - \frac{1}{4}y^2$ (you have to use integration by parts for both antiderivatives).

*b)* $\vec{F} = x(\ln(x^2)+1)\sin(y)\vec{\imath} + x^2\ln(x)\cos(y)\vec{\jmath}$

**Solution.** The only issue may come from $\ln(x)$, which is only defined for $x > 0$. So the region of definition is $R = \{\,(x,y) \,:\, x > 0\,\}$, which is the right half plane (so is simply connected). Now let's use the scalar curl test: we see that

$$\frac{\partial F_2}{\partial x} = 2x\ln(x)\cos(y) + x^2\frac{1}{x}\cos(y), \qquad \frac{\partial F_1}{\partial y} = x(\ln(x^2)+1)\cos(y)$$

are equal, so by the scalar curl test (since the region of definition is simply connected), $\vec{F}$ has a potential function on $R$. Now we have to find one. We're looking for $f$ defined on $R$, so that $\nabla f = \vec{F}$, i.e.

$$\frac{\partial f}{\partial x} = x(\ln(x^2)+1)\sin(y), \qquad \frac{\partial f}{\partial y} = x^2\ln(x)\cos(y)$$

Integrating (with respect to $y$) the second equation, we find that

$$f(x,y) = \int x^2 \ln(x)\cos(y)\,dy = x^2\ln(x)\sin(y) + \phi(x)$$

where $\phi(x)$ is the "constant of integration" with respect to $y$, which is just a function of $x$ that we are trying to find to satisfy the first equation above. Now taking the partial with respect to $x$ of this function gives,

$$\frac{\partial f}{\partial x} = 2x\ln(x)\sin(y) + x\sin(y) + \phi'(x) = x(2\ln(x)+1)\sin(y) + \phi'(x)$$

comparing with what we want (and using $\ln(x^2) = 2\ln(x)$), we see that $\phi'(x) = 0$, so $\phi(x)$ is a constant, which we might as well take to be zero. So finally, we have found $f(x,y) = x^2\ln(x)\sin(y)$ to be a potential function for $\vec{F}$ on the region $R$.

*c)* $\vec{F} = 2xy^2\vec{\imath} + x^2\vec{\jmath}$

**Solution.** Here $\vec{F}$ is defined on all of $\mathbb{R}^2$, which is simply connected. First trying the scalar curl test, we calculate

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2x - 4xy \neq 0$$

so by the scalar curl test, $\vec{F}$ is not path-independent and does not have a potential function on $\mathbb{R}^2$.

*d)* $\vec{F} = (y^2 + y + 2xy)\vec{\imath} + (x^2 + x + 2xy)\vec{\jmath}$

**Solution.** This vector field is defined everywhere (i.e. on all of $\mathbb{R}^2$) so has simply connected region of definition. Let's use the scalar curl test:

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2x + 1 + 2y - (2y + 1 + 2x) = 0$$

so since $\vec{F}$ is defined on a simply connected region, by the scalar curl test, $\vec{F}$ has a potential function and is path-independent on $\mathbb{R}^2$. Let's find a potential function. We're looking for $f$, so that $\nabla f = \vec{F}$, i.e.

$$\frac{\partial f}{\partial x} = y^2 + y + 2xy, \quad \frac{\partial f}{\partial y} = x^2 + x + 2xy.$$

Integrating (with respect to $x$) the first equation, we find that

$$f(x,y) = \int (y^2 + y + 2xy)dx = xy^2 + xy + x^2y + \phi(y)$$

where $\phi(x)$ is the "constant of integration" with respect to $x$, which is just a function of $y$ that we are trying to find to satisfy the second equation above. Now taking the partial with respect to $y$ of this function gives,

$$\frac{\partial f}{\partial y} = 2xy + x + x^2 + \phi'(y)$$

comparing with what we want, we see that $\phi'(x) = 0$, so $\phi(x)$ is a constant, which we might as well take to be zero. So finally, we have found $f(x,y) = xy^2 + xy + x^2y$ to be a potential function for $\vec{F}$ on $\mathbb{R}^2$.

*e)* $\vec{F} = \dfrac{1+y}{y^2 + 2y + 2 - 2x + x^2}\vec{\imath} + \dfrac{1-x}{y^2 + 2y + 2 - 2x + x^2}\vec{\jmath}$

**Solution.** Here, $\vec{F}$ is not defined whenever either of the denominators is zero. Cleaning up the denominator we find

$$y^2 + 2y + 2 - 2x + x^2 = (y^2 + 2y + 1) + (x^2 - 2x + 1) = (y+1)^2 + (x-1)^2$$

which is a sum of squares, so is only zero when both terms are zero, i.e. only at $(x, y) = (-1, 1)$. So the region $R$ of definition of $\vec{F}$ is the plane removed the point $(1, -1)$, which is *not* a simply connected region! Let's see if the scalar curl test can easily rule out path-independence:

$$\frac{\partial F_2}{\partial x} = \frac{-((y+1)^2 + (x-1)^2) - 2(x-1)(1-x)}{((y+1)^2 + (x-1)^2)^2} = \frac{(x-1)^2 - (y+1)^2}{(y+1)^2 + (x-1)^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{((y+1)^2 + (x-1)^2) - 2(1+y)(y+1)}{((y+1)^2 + (x-1)^2)^2} = \frac{(x-1)^2 - (y+1)^2}{(y+1)^2 + (x-1)^2}$$

these are equal, so the scalar curl is zero, but the region of definition is not simply connected. We're out of luck! The scalar curl test can't help us at all! But it looks suspiciously like another vector field we know very well that has zero scalar curl, but yet is not path-independent! So if we still want to try to disprove path-independence, we have to find a closed curve with non-zero line integral. Since the "action" seems to be happening around the point $(1, -1)$, let's just try a circle of radius 1 around that point, which is parameterized by $\gamma(t) = (1 + \cos(t), -1 + \sin(t))$ for $0 \le t \le 2\pi$. Computing the line integral, we find

$$\int_\gamma \vec{F} = \int_0^{2\pi} \left( \frac{\sin(t)\vec{\imath} - \cos(t)\vec{\jmath}}{\sin^2(t) + \cos^2(t)} \right) \cdot (-\sin(t)\vec{\imath} + \cos(t)\vec{\jmath})\, dt = -\int_0^{2\pi} dt = -2\pi \neq 0$$

so by the closed curve test, $\vec{F}$ is *not* path-independent.

*f)* $\vec{F} = \dfrac{x}{x^2 + y^2}\vec{\imath} + \dfrac{y}{x^2 + y^2}\vec{\jmath}$

**Solution.** The region of definition is anywhere the denominator doesn't vanish, i.e. the whole plane removed the origin, so *not* a simply connected region. Let's try the scalar curl first:

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = y\frac{2x}{(x^2 + y^2)^2} - x\frac{2y}{(x^2 + y^2)^2} = 0$$

so we're out of luck! Since the "hole" in the region of definition is at the origin, let's try a line integral over a circle around the origin, $\gamma(t) = (\cos(t), \sin(t))$ for $0 \le t \le 2\pi$. Computing, we find

$$\int_\gamma \vec{F} = \int_0^{2\pi} \left( \frac{\cos(t)\vec{\imath} + \sin(t)\vec{\jmath}}{\sin^2(t) + \cos^2(t)} \right) \cdot (-\sin(t)\vec{\imath} + \cos(t)\vec{\jmath})\, dt = \int_0^{2\pi} 0 dt = 0$$

which is still inconclusive, though provides some evidence that actually a potential function exists on the whole region of definition. Let's try to find one.

As before, we're looking for a function $f$ defined on the plane removed the origin, so that

$$\frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2}$$

Integrating the first equation with respect to $x$ yields

$$f(x, y) = \int \frac{x}{x^2 + y^2}\, dx = \frac{1}{2}\ln(x^2 + y^2) + \phi(y)$$

then taking the derivative with respect to $y$, from the second equation above, we want

$$\frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2} + \phi'(y) = \frac{y}{x^2 + y^2}$$

so that we need $\phi'(y) = 0$, so $\phi$ is constant, and we might as well take $\phi(y) = 0$. In total, $\vec{F}$ has potential function $f(x, y) = \frac{1}{2}\ln(x^2 + y^2)$, which actually is defined on the whole plane remove the origin.

**3.** For the following vector fields $\vec{F}$ and curves $\gamma$, calculate the line integral $\int_\gamma \vec{F}$.

*a)* $\vec{F} = x^2\vec{\imath} + y^2\vec{\jmath}$ and $\gamma$ is the line segment starting at the origin and ending at the point $(2, 3)$.

**Solution.** Since $\vec{F}$ is separable we know it is a gradient field, with potential function given by the sum of the antiderivatives $f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$. By the FTC, for any $\gamma$ from the origin to $(2, 3)$, we have

$$\int_\gamma \vec{F} = f(2, 3) - f(0, 0) = \frac{1}{3}(2^3 + 3^3) - 0 = \frac{35}{3}.$$

*b)* $\vec{F} = (x^2 + y)\vec{\imath} + y^2\vec{\jmath}$ and $\gamma$ is the line segment starting at the origin and ending at the point $(2, 3)$.

**Solution.** The scalar curl of $\vec{F}$ is $-1 \neq 0$, so $\vec{F}$ is *not* a gradient field, so we can't use the FTC, and so must calculate the integral directly. Let $\gamma(t) = (2t, 3t)$ for $0 \leq t \leq 1$, then

$$\int_\gamma \vec{F} = \int_0^1 ((4t^2 + 3t)\vec{\imath} + 9t^2\vec{\jmath}) \cdot (2\vec{\imath} + 3\vec{\jmath})\, dt = \int_0^1 (35t^2 + 6t)\, dt = \frac{44}{3}.$$

*c)* $\vec{F} = (y + e^{\sin(x)})\vec{\imath} + (x + \sqrt{y^2 + 1})\vec{\jmath}$ and $\gamma$ is the unit circle going counter clockwise

**Solution.** Staring at this vector field, we can see that if it's not a gradient field, then the direct line integral computation is going to be hard! So let's hope it's a gradient field. Then the line integral around any closed curve is 0. Here are two ways to go about it.

First, you might notice that, while the variables are not separated, some parts are, i.e. we can write $\vec{F}$ as a sum of two vector fields

$$\vec{F}(x, y) = y\vec{\imath} + x\vec{\jmath} \; + \; e^{\sin(x)}\vec{\imath} + \sqrt{y^2 + 1}\vec{\jmath}$$

the second is separated so we know is a gradient field, and the first, we can just spot the potential function as $f(x, y) = xy$. The sum of gradient fields is a gradient field, so indeed, $\vec{F}$ is a gradient field.

You might just go right for it using the algorithm discussed in class, but one issue you'll run into is that the functions $e^{\sin x}$ and $\sqrt{y^2 + 1}$ do not have nice antiderivatives.

Second, you might use the scalar curl test. In this case,

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - 1 = 0$$

so we just need to check the region of definition. The only bad thing that could happen is under the square root, but $y^2 + 1$ is always positive, so this never causes a problem. Thus $\vec{F}$ is defined on the whole plane! In particular, the region of definition is simply connected, so the scalar curl test applies, and shows that $\vec{F}$ is a gradient field.

*d)* $\vec{F} = \dfrac{1}{1 + y^2 + 2xy + x^2}\vec{\imath} + \dfrac{1}{1 + y^2 + 2\,xy + x^2}\vec{\jmath}$ and $\gamma$ is the unit circle going counter clockwise.

**Solution.** You can compute that the scalar curl is zero! Also, note that, cleaning up the denominator $1 + y^2 + 2xy + x^2 = 1 + (x + y)^2$, it is always positive. So $\vec{F}$ is defined everywhere.

So by the scalar curl test, $\vec{F}$ is path-independent, hence any line integral around a closed curve is zero.

If you did not realize this, you could compute by hand with the standard parameterization of the unit circle. Then:

$$\begin{aligned}\int_\gamma \vec{F} &= \int_0^{2\pi} \left(\frac{\vec{\imath}+\vec{\jmath}}{1+(\cos(t)+\sin(t))^2}\right)\cdot(-\sin(t)\vec{\imath}+\cos(t)\vec{\jmath})\,dt \\ &= \int_0^{2\pi} \frac{-\sin(t)+\cos(t)}{1+(\cos(t)+\sin(t))^2}\,dt = \int_1^1 \frac{1}{1+u^2}\,du = 0\end{aligned}$$

using the substitution $u=\cos(t)+\sin(t)$, so that $du=(-\sin(t)+\cos(t))dt$, $u(0)=1$ and $u(2\pi)=1$. We can also see that this integral is zero because the integrand on the interval $[0,\pi]$ is exactly it's own negative on the interval $[\pi,2\pi]$, so everything cancels over the interval $[0,2\pi]$.

*e)* $\vec{F}=(x^2+y^2)\vec{\imath}+xy\vec{\jmath}$ and $\gamma$ is the unit circle going counter clockwise.

**Solution.** The scalar curl is $-y\neq 0$, so $\vec{F}$ is not a gradient field. So we compute by hand with the standard parameterization of the unit circle:

$$\begin{aligned}\int_\gamma \vec{F} &= \int_0^{2\pi} ((\cos^2(t)+\sin^2(t))\vec{\imath}+\sin(t)\cos(t)\vec{\jmath})\cdot(-\sin(t)\vec{\imath}+\cos(t)\vec{\jmath})\,dt \\ &= \int_0^{2\pi}(-\sin(t)+\sin(t)\cos^2(t))\,dt = -\cos(t)-\frac{1}{3}\cos^3(t)\Big|_0^{2\pi}=0.\end{aligned}$$

**4.** Let $\vec{F}(x,y)=\vec{\imath}+\cos(x)\vec{\jmath}$ be a vector field on $\mathbb{R}^2$.

*a)* Is $\vec{F}$ path-independent? State which test you are using.

*b)* Let $\gamma(t)=(t,0)$ and $\delta(t)=(t,\sin(t))$, for $0\le t\le\pi$. Compute the line integrals $\int_\gamma \vec{F}$ and $\int_\delta \vec{F}$. Does your calculation reaffirm your answer to the previous part?

**Solution.**

*a)* The scalar curl of $\vec{F}$ is $-\sin(x)\neq 0$. So by the scalar curl test, $\vec{F}$ is *not* path-independent.

*b)* Compute:

$$\begin{aligned}\int_\gamma \vec{F} &= \int_0^\pi (\vec{\imath}+\cos(t)\vec{\jmath})\cdot\vec{\imath}\,dt = \int_0^\pi dt = \pi \\ \int_\delta \vec{F} &= \int_0^\pi (\vec{\imath}+\cos(t)\vec{\jmath})\cdot(\vec{\imath}+\cos(t)\vec{\jmath})\,dt = \int_0^\pi (1+\cos^2(t))\,dt = \frac{3\pi}{2}\end{aligned}$$

Now both curves start at the origin and end at $(\pi,0)$, but have different line integrals, affirming the fact that $\vec{F}$ is not path-independent.

**5.** Match the integral with the appropriate region of integration:

*a)* The triangle with vertices $(0,0),(2,0),(0,1)$.
*b)* The triangle with vertices $(0,0),(0,2),(1,0)$.
*c)* The triangle with vertices $(0,0),(2,0),(2,1)$.
*d)* The triangle with vertices $(0,0),(1,0),(1,2)$.

*A)* $\int_{x=0}^1\int_{y=0}^{2-2x} f(x,y)\,dydx$
*B)* $\int_{y=0}^1\int_{x=0}^{2-2y} f(x,y)\,dxdy$
*C)* $\int_{x=0}^1\int_{y=0}^{2x} f(x,y)\,dydx$
*D)* $\int_{y=0}^1\int_{x=2y}^{2} f(x,y)\,dxdy$

**Solution.** Matching: *a)-B); b)-A); c)-D); d)-C).* Switching the order of integration in each integral.

*A)* $\int_{y=0}^{2}\int_{x=0}^{1-\frac{1}{2}y} f(x,y)\,dxdy$
*B)* $\int_{x=0}^{2}\int_{y=0}^{1-\frac{1}{2}x} f(x,y)\,dydx$
*C)* $\int_{y=0}^{2}\int_{x=\frac{1}{2}y}^{1} f(x,y)\,dxdy$
*D)* $\int_{x=0}^{2}\int_{y=0}^{\frac{1}{2}x} f(x,y)\,dydx$

**6.** Calculate the double integral

$$\int_{x=-1}^{1}\int_{y=0}^{1-x^2} \sin\big(\pi(1-y)^{3/2}\big)\,dydx.$$

Hint: you may need to switch the order of integration.

**Solution.** Since the inner integral (in terms of $y$) seems to be impossible, let's try switching the order of integration. Drawing the shape involved, we see that it's the region between the downward facing parabola $y = 1 - x^2$ and the $x$-axis. Solving for the equation of the parabola for $x$ in terms of $y$, we get $x = \pm\sqrt{1-y}$. So the reparameterization of the region is now $\{(x,y) : 0 \le y \le 1,\ -\sqrt{1-y} \le x \le \sqrt{1-y}\}$. The integral is now

$$\begin{aligned}\int_{y=0}^{1}\int_{x=-\sqrt{1-y}}^{\sqrt{1-y}} \sin\big(\pi(1-y)^{3/2}\big)\,dxdy &= \int_{y=0}^{1} \sin\big(\pi(1-y)^{3/2}\big)\,x\Big|_{x=-\sqrt{1-y}}^{\sqrt{1-y}}\,dy \\ &= \int_{y=0}^{1} 2\sqrt{1-y}\,\sin\big(\pi(1-y)^{3/2}\big)\,dy.\end{aligned}$$

At this point we have to use $u$-substitution. Letting $u = \pi(1-y)^{3/2}$ then $du = -\frac{3\pi}{2}\sqrt{1-y}dy$, and the limits range from $u = \pi$ to $u = 0$. Thus the integral now becomes

$$-\frac{4}{3\pi}\int_{u=\pi}^{0} \sin(u)\,du = \frac{4}{3\pi}\cos(u)\Big|_{u=\pi}^{0} = \frac{8}{3\pi}.$$

**7.** Calculate the volume of the region under the plane $3x + 4y + 6z = 12$ and in the first octant, i.e. where $x \ge 0$, $y \ge 0$, and $z \ge 0$.

**Solution.** Let's set this up as a double integral. Viewing the plane $3x+4y+6z = 12$ as the graph of the function $f(x,y) = 2 - \frac{1}{2}x - \frac{2}{3}y$ (just by solving for $z$), then the volume will be the integral $\int_R f$, where $R$ is the region in the $x$-$y$-plane under the graph of $f$ and in the "first quadrant" (i.e. $x \ge 0$ and $y \ge 0$). The region $R$ is bounded by the $x$- and $y$-axis along with the intersection of the plane with the $x$-$y$-plane (i.e. $z = 0$), which is the line $3x + 4y = 12$. So $R$ is the triangle with verticles $(0,0)$, $(4,0)$, and $(0,3)$. We can thus set up the integral as a double integral

$$\begin{aligned}\int_R f = \int_{x=0}^{4}\int_{y=0}^{3-\frac{3}{4}x} (2 - \frac{1}{2}x - \frac{2}{3}y)\,dydx &= \int_{x=0}^{4} (2y - \frac{1}{2}xy - \frac{1}{3}y^2)\Big|_{y=0}^{3-\frac{3}{4}x}\,dx \\ = \int_{x=0}^{4}\big(2(3-\frac{3}{4}x) - \frac{1}{2}x(3-\frac{3}{4}x) - \frac{1}{3}(3-\frac{3}{4}x)^2\big)\,dx &= \int_{x=0}^{4}(3 - \frac{3}{2}x + \frac{3}{16}x^2)\,dx = 4.\end{aligned}$$

**8.** Calculate the volume of the region under the graph of $f(x,y) = 1 - 8x - y^2$ and bounded by the $x$-$y$-plane and the $y$-$z$-plane (i.e. where $x \ge 0$ and $z \ge 0$).

**Solution.** Let's set this up as a double integral. We have to figure out the region $R$ in the $x$-$y$-plane under the graph of $f$. The intersection of the graph of $f$ with the $x$-$y$-plane is given by the equation $0 = 1 - 8x - y^2$. Rewriting this as $x = \frac{1}{8} - \frac{1}{8}y^2$ shows that this is a parabola opening up to the left with it's "peak" at the point $(\frac{1}{8}, 0)$. Since we are only considering the part bounded by the $y$-$z$-plane, the region $R$ is to the right of the $y$-axis. The parabola intersects the $y$-axis at the points $(0, -1)$ and $(0, 1)$. So we can set up the integral as

$$\int_R f = \int_{y=-1}^{1} \int_{x=0}^{\frac{1}{8}-\frac{1}{8}y^2} (1 - 8x - y^2)\, dxdy = \int_{y=-1}^{1} (x - 4x^2 - y^2x)\Big|_{x=0}^{\frac{1}{8}-\frac{1}{8}y^2} dy$$
$$= \int_{y=-1}^{1} \big((\frac{1}{8} - \frac{1}{8}y^2) - 4(\frac{1}{8} - \frac{1}{8}y^2)^2 - y^2(\frac{1}{8} - \frac{1}{8}y^2)\big)\, dy = \int_{y=-1}^{1} \frac{1}{16}(1 - 2y^2 + y^4)\, dy = \frac{1}{15}.$$

**9.** Calculate the double integral $\int_R f$, where $f(x, y) = 8xy$ and $R$ is the region (in the plane) between circles of radius 1 and 2 in the first quadrant. Hint: you may want to use polar coordinates.

**Solution.** Let's take the hint and use polar coordinates (anyway this makes sense since we're integration over an annulus). The region $R$ in the first quadrant between the two circles is parameterized in polar coordinates by $\{\,(x, y) \,:\, 0 \le \theta \le \pi/2,\ 1 \le r \le 2\,\}$. The function $f$ is given in polar coordinates by $f(r\cos(\theta), r\sin(\theta)) = 8 \cdot r\cos(\theta) \cdot r\sin(\theta) = 8r^2\cos(\theta)\sin(\theta)$. Thus the integral is given by

$$\int_R f = \int_{\theta=0}^{\pi/2} \int_{r=1}^{2} 8r^2\cos(\theta)\sin(\theta)\, r\, drd\theta = 15.$$

In doing the above integral over $\theta$, it might be helpful to use the "double angle" formula $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$.

**10.** Calculate the double integral $\int_R f$, where $f(x, y) = e^{x^2+y^2}$ and $R$ is the unit disk centered at the origin.

**Solution.** Again, because we are integration over a circle (and since the integrand has $x^2 + y^2$ in it) we should use polar coordinates. In the case, the unit disk is $\{\,(x, y) \,:\, 0 \le \theta \le 2\pi,\ 0 \le r \le 1\,\}$ and the function becomes $f(r\cos(\theta), r\sin(\theta)) = e^{r^2}$. Then the integral is given by

$$\int_R f = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} e^{r^2}\, r\, drd\theta = 2\pi \int_{r=0}^{1} e^{r^2}\, r\, dr = \pi e^{r^2}\Big|_{r=0}^{1} = \pi(e - 1).$$

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