Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Spring 2012

Problem Set # 3 (Fri 10 Feb 2012) Selected Solutions

1. CM 14.4

- Exercise 14

  Solution. $\nabla z|_{(x,y)} = \frac{1}{(x+y)^2}(ye^y\vec{\boldsymbol{\imath}} + e^y(x^2 + xy - x)\vec{\boldsymbol{\jmath}})$

- Exercise 24

  Solution. $\nabla f|_{(1,2)} = 2\vec{\boldsymbol{\imath}} + 13\vec{\boldsymbol{\jmath}}$ and $f_{\vec{u}}(1,2) = -46/5$.

- Exercise 30-35

  Solution.
  30 From $(-2,2)$ in the $\vec{\boldsymbol{\imath}}$ direction the values of f are decreasing, so the directional derivative is negative.
  31 From $(0,-2)$ in the $\vec{\boldsymbol{\jmath}}$ direction the values of f are decreasing, so the directional derivative is negative.
  32 From $(-1,1)$ in the $\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}}$ direction, we're moving along a contour of f, so the values aren't changing, so the directional derivative is zero.
  33 From $(-1,1)$ in the $-\vec{\boldsymbol{\imath}}+\vec{\boldsymbol{\jmath}}$ direction the values of f are increasing, so the directional derivative is positive.
  34 From $(0,-2)$ in the $\vec{\boldsymbol{\imath}}+2\vec{\boldsymbol{\jmath}}$ direction the values of f are decreasing, so the directional derivative is negative.
  35 From $(0,-2)$ in the $\vec{\boldsymbol{\imath}}-2\vec{\boldsymbol{\jmath}}$ direction the values of f are increasing, so the directional derivative is positive.

- Problem 66

  Solution. In fact $\|\nabla f|_P\|$ is larger. Where the level curves are closer together, it means that the function is increasing faster.

- Problem 70

  Solution. First, $\nabla f|_{(-1,2)} = -16\vec{\boldsymbol{\imath}} + 12\vec{\boldsymbol{\jmath}}$. By the properties of the gradient, a vector in the direction of maximum change is $\nabla f|_{(-1,2)}$ while a vector in the direction of minimum change is its negative. A vector in the direction of no change is any vector perpendicular to $\nabla f|_{(-1,2)}$, for example $3\vec{\boldsymbol{\imath}} + 4\vec{\boldsymbol{\jmath}}$.

2. CM 14.5

- Exercise 12

  Solution. $\nabla f = \frac{2xy}{x^2+5}\vec{\boldsymbol{\imath}} + \ln(x^2+5)\vec{\boldsymbol{\jmath}} + 2ze^{z^2}\vec{\boldsymbol{k}}$

- Exerciese 18

Solution. $\nabla f = y\cos(xy)\vec{\imath} + (x\cos(xy) + z\cos(yz))\vec{\jmath} + y\cos(yz)\vec{k}$ and so $\nabla f|_{(1,\pi,-1)} = -\pi\vec{\imath} - \pi\vec{k}$.

- Problem 40

Solution. Since this surface is a level surface of $F(x,y,z) = 3x^2 - 4xy + z^2$, a vector normal to the tangent plane at the point $(a,a,a) \in \mathbb{R}^3$ (which by the way, you should verify is actually on the surface!) is $\nabla F|_{(a,a,a)} = 2a\vec{\imath} - 4a\vec{\jmath} + 2a\vec{k}$, so that an equation for the plane is $2a(x-a) - 4a(y-a) + 2a(z-a) = 0$, which simplifies to $2ax - 4ay + 2az = 0$. Since $a \neq 0$, this further simplifies to $2x - 4y + 2z = 0$. In particular, this plane doesn't depend on a and goes through the origin! This surface is a *cone*.

- Problem 42

Solution. This surface is the level surface $F(x,y,z) = 7$ of the function $F(x,y,z) = x^2 + y^2 - xyz$. Hence a normal vector to the surface at the point $(2,3,1)$ is given by

$$\nabla F|_{(2,3,1)} = \big((2x - yz)\vec{\imath} + (2y - xz)\vec{\jmath} - xy\vec{k}\big)|_{(2,3,1)} = \vec{\imath} + 4\vec{\jmath} - 6\vec{k}.$$

By the point/normal equation for a plane, the tangent plane is then

$$x + 4y - 6z = 1 \cdot 2 + 4 \cdot 3 - 6 \cdot 1 = 8.$$

Solving for z, we get

$$z = \frac{x^2 + y^2 - 7}{xy}$$

which we can do as long as neither x nor y is zero, which is the case at our point $(2,3,1)$. Thus the surface is (at least at our point) the graph of the function $f(x,y) = \frac{x^2+y^2-7}{xy}$. Hence a normal vector to the surface at the point $(2,3,1)$ is given by

$$\nabla f|_{(2,3)} - \vec{k} = \left(\frac{2x^2y - y(x^2 + y^2 - 7)}{x^2y^2}\vec{\imath} + \frac{2xy^2 - x(x^2 + y^2 - 7)}{x^2y^2}\vec{\jmath}\right)\Big|_{(2,3)} - \vec{k} = \frac{1}{6}\vec{\imath} + \frac{2}{3}\vec{\jmath} - \vec{k}.$$

By the point/normal equation for a plane, the tangent plane is then

$$\frac{1}{6}x + \frac{2}{3}y - z = \frac{1}{6} \cdot 2 + \frac{2}{3} \cdot 3 - 1 \cdot 1 = \frac{4}{3}.$$

This is the same plane as above (just multiply through by 6). What way was easier?

- Problem 48

Solution. The surface $z = 20 - (2x^2 + y^2)$ is the graph of the function $f(x,y) = 20 - (2x^2 + y^2)$. The direction of greatest ascent of the graph of f above the point $(1,3)$ is given by the gradient

$$\nabla f|_{(1,3)} = -4\vec{\imath} - 6\vec{\jmath}.$$

Hence the direction of greatest descent is $-\nabla f|_{(1,3)} = 4\vec{\imath} + 6\vec{\jmath}$. The slope of this descent is the rate of change of f in this direction, which is the directional derivative in the direction $\vec{v} = 4\vec{\imath} + 6\vec{\jmath}$. To calculate the directional derivative, we need a unit vector in this direction, which is given by $\vec{u} = \frac{1}{\|v\|}v = \frac{1}{\sqrt{13}}(2\vec{\imath} + 3\vec{\jmath})$. Finally, the directional derivative is

$$f_{\vec{u}}(1,3) = \nabla f|_{(1,3)} \cdot \vec{u} = (-4\vec{\imath} - 6\vec{\jmath}) \cdot \frac{1}{\sqrt{13}}(2\vec{\imath} + 3\vec{\jmath}) = -\frac{1}{\sqrt{13}}26 = -2\sqrt{13}.$$

Another way to do this is to realize that the slope you'll follow as you start moving in the direction of fastest descent will be negative of the slope in the direction of fastest ascent. By the properties of the gradient listed on page 752, the slope (i.e. rate) of fastest ascent is the magnitude of the gradient. Thus the slope we are looking for is:

$$-\|\nabla f|_{(1,3)}\| = -\sqrt{4^2+6^2} = -\sqrt{52},$$

which is the same as we got above!

- Problem 54

Solution. The surface in question is the graph of the function $f(x,y) = 1 + x^2 + y^2$. At the point $(a, b, 1 + a^2 + b^2)$ on the surface, the tangent plane has normal vector

$$\nabla f|_{(a,b)} - \vec{\boldsymbol{k}} = 2a\vec{\boldsymbol{\imath}} + 2b\vec{\boldsymbol{\jmath}} - \vec{\boldsymbol{k}}.$$

Two planes are parallel if their normal vectors are proportional. The normal vector to the plane in part *a)* is $\vec{\boldsymbol{k}}$, so our tangent plane normal vector is proportional to it only above the point $(a, b) = (0, 0)$, i.e. at the point $(0, 0, 1)$

Rewriting the plane in part *b)* as $6x - 10y - z = -5$, we see its normal vector is $6\vec{\boldsymbol{\imath}} - 10\vec{\boldsymbol{\jmath}} - \vec{\boldsymbol{k}}$. OK, so we're looking for points $(a, b, f(a, b))$, so that

$$2a\vec{\boldsymbol{\imath}} + 2b\vec{\boldsymbol{\jmath}} - \vec{\boldsymbol{k}} = \lambda(6\vec{\boldsymbol{\imath}} - 10\vec{\boldsymbol{\jmath}} - \vec{\boldsymbol{k}})$$

for some nonzero number λ (i.e. making the normal vectors proportional). So you get three equations in three unknowns

$$2a = 6\lambda, \quad 2b = -10\lambda, \quad -1 = -\lambda.$$

From the third equation, $\lambda = 1$ is forced. But then that forces $a = 3$ and $b = -5$ in the first two equations! This means that the only point where this happens is $(a, b, f(a, b)) = (3, -5, 35)$.

3. CM 17.1

- Exercise 16

Solution. You want a line going through the points $(3, -2, 2)$ and $(0, 2, 0)$. Remember that a line going through points P and Q is given by $\gamma(t) = P + \overrightarrow{PQ}t$. In this case, it's $\gamma(t) = (3 - 3t, -2 + 4t, 2 - 2t)$, for example.

- Exercise 22

Solution. For example, $\gamma(t) = (0, 3cos(t), 2 + 3\sin(t))$ works.

- Problem 46

Solution. Given a plane with normal vector $\vec{n}$ at a point P, the point/normal equation says "the plane consists of points whose displacement vector from P is perpendicular to $\vec{n}$". So if at the same time you want to be on two planes, with normal vectors $\vec{n}_1$ and $\vec{n}_2$ at the *same point* P, then your displacement vector from P should be perpendicular to *both* $\vec{n}_1$ and $\vec{n}_2$. But since $\vec{n}_1 \times \vec{n}_2$ is perpendicular to both, its head is on the intersection line. I also discussed this issue in lecture.

- Problem 56

Solution. The point is given to you $(-4, 2, 3)$, a direction vector fitting their description is $\vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}}$. Then you can parameterize the line.

4. CM 17.2

- Problem 10. Let $\gamma(t) = ((t-1)^2, 2, 2t^3 - 3t^2)$. When does $\gamma(t)$ stop?

  Solution. "Stops" means "has speed 0". So we compute the speed at time t,

  $$\|\vec{\gamma}'(t)\| = \|2(t-1)\vec{\boldsymbol{\imath}} + (6t^2 - 6t)\vec{\boldsymbol{k}}\| = \sqrt{4(t-1)^2 + 36t^2(t-1)^2}$$

  which is equal to 0 when $(t-1)^2(4 + 36t^2) = 0$, which is only when $t = 1$.